THE REVIEW OF SYMBOLIC LOGIC
Volume 7, Number 3, September 2014

UNIFORM INTERPOLATION IN SUBSTRUCTURAL LOGICS

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Abstract. Uniform interpolation property of a given logic is a stronger form of Craig's interpolation property where both *pre-interpolant* and *post-interpolant* always exist uniformly for any provable implication in the logic. It is known that there exist logics, e.g., modal propositional logic **S4**, which have Craig's interpolation property but do not have uniform interpolation property. The situation is even worse for predicate logics, as classical predicate logic does not have uniform interpolation property as pointed out by L. Henkin.

In this paper, uniform interpolation property of basic substructural logics is studied by applying the proof-theoretic method introduced by A. Pitts (Pitts, 1992). It is shown that uniform interpolation property holds even for their predicate extensions, as long as they can be formalized by sequent calculi without contraction rules. For instance, uniform interpolation property of full Lambek predicate calculus, i.e., the substructural logic without any structural rule, and of both linear and affine predicate logics without exponentials are proved.

§1. Preliminaries.

Uniform interpolation property is a stronger form of Craig interpolation property. Craig interpolation property of a given logic **L** says that for any formula α and β, if $\alpha \to \beta$ is provable in **L** then there exists a formula γ such that both $\alpha \to \gamma$ and $\gamma \to \beta$ are provable in **L** and moreover $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$ holds. Here $V(\varphi)$ denotes the set of all propositional variables in a given formula φ. Such a formula γ is called an *interpolant* of the formula $\alpha \to \beta$. Even if Craig interpolation property holds in a logic **L**, an interpolant of a given formula $\alpha \to \beta$ is not always determined uniquely, and depends in general on both formulas α and β. On the other hand, *uniform interpolation property* of a logic **L** means that for any formula of the form $\alpha \to \beta$ which is provable in **L** there exist an interpolant which determines only by α (*post-interpolant*) and also an interpolant which determines only by β (*pre-interpolant*). For its precise definition, see Section 3.

Received: July 17, 2013.

doi:10.1017/S175502031400015X

It is easy to show that classical propositional logic has uniform interpolation property. But to show it for intuitionistic propositional logic was never trivial. This was shown first by using proof theoretic way in Pitts (1992) and then semantically in Ghilardi & Zawadowski (1995a) and Visser (1996). On the other hand, uniform interpolation property holds neither for classical predicate logic nor for intuitionistic predicate logic, as shown in Henkin (1963). (See also an interesting note in van Benthem, 2008 on what Craig thought at first.) There exist also modal propositional logics, for example **S4**, which have Craig's interpolation property but do not have uniform interpolation property (Ghilardi & Zawadowski, 1995b). Uniform interpolation property for some modal propositional logics has been shown proof theoretically or semantically. See, e.g., Bílková (2007), Visser (1996), and also D'Agostino (2008) for general information.

In the present paper, we will study uniform interpolation property of substructural logics by using the proof theoretic method introduced by Pitts (Pitts, 1992). To apply the method, it is necessary to formalize a logic under consideration in a cut-free sequent calculus. A key proposition (Proposition 5) of Pitts (1992) is shown by using the induction of the *weight* of formulas and of multisets of formulas, considering *backward proof searches* (see, e.g., Bílková, 2007). Here, the weight of a formula means roughly its complexity. A main obstacle in this argument will be caused by such a rule that the weight of its upper sequent(s) is not smaller than that of the lower sequent. Typically, this can be caused by contraction rules. To resolve this difficulty, Pitts (1992) employed a *contraction-free* sequent calculus for intuitionistic propositional logic introduced by Dyckhoff (1992) and Hudelmaier (1989), instead of the original Gentzen's calculus **LJ**. Here a "contraction-free" calculus means that standard contraction rules are not contained explicitly. (Note that contraction rule seems to be eliminated from the calculus for intuitionistic propositional logic in Table 3 in Pitts, 1992, but the above difficulty is shifted to the rule ($\rightarrow\Rightarrow$). Thus further elaborations were necessary, as shown in Dyckhoff, 1992; Pitts, 1992.) Similarly, to show uniform interpolation property of modal propositional logics **K** and **T**, Bílková (2007) used contraction-free calculi for them which were introduced in Heuerding (1998).

From these observations, we can expect that uniform interpolation property may hold for substructural logics without contraction rules. In this paper, we develop a comprehensive study of uniform interpolation property of these logics, and show that uniform interpolation property holds even for predicate extensions of basic substructural logics without contraction rules, i.e., **QFL**, $\mathbf{QFL_e}$, $\mathbf{QFL_{ew}}$, $\mathbf{QInFL_e}$ and $\mathbf{QInFL_{ew}}$. The first is known as full Lambek predicate calculus, and the last two as linear and affine predicate logic without exponentials, respectively.

Our paper is organized as follows. After introducing cut-free sequent calculi for basic substructural propositional logics in the next section, we show in Section 3 uniform interpolation property of substructural propositional logics $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$. Both of them are *commutative*, i.e., having exchange rule, and the latter has weakening rule in addition. These results are extended in Section 4 to their involutive extensions $\mathbf{InFL_e}$ (i.e., **MALL**) and $\mathbf{InFL_{ew}}$, which are obtained from $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$ by adding the law of double negation $\neg\neg\varphi \rightarrow \varphi$. As these arguments rely on commutativity, a certain modification will be needed in order to deal with the case for substructural propositional logics **FL** without any structural rules. This can be carried out in Section 5, and uniform interpolation property of **FL** is shown. In the last section, we extend these results for predicate logics. Thus, we have uniform interpolation for predicate extensions of all basic substructural propositional logics mentioned above. This makes an interesting contrast with negative result by Henkin for classical and intuitionistic predicate logics. For general information on substructural logics, see Galatos *et al.* (2007).

§2. Sequent calculi for basic substructural propositional logics. In this section, we will introduce sequent calculi for three basic substructural propositional logics **FL**, $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$. We will use **FL**, $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$ etc. both for each of the calculus and for the set of provable formulas in it. Thus, we say, for instance, that a logic is *substructural* over **FL** if it is an axiomatic extension of **FL**. The calculus **FL** has no structural rules, which is called *full Lambek propositional calculus*, and is regarded as the basis for substructural propositional logics. Both $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$ are substructural logics over **FL**, where $\mathbf{FL_e}$ ($\mathbf{FL_{ew}}$) can be obtained from **FL** by adding exchange rule (exchange and weakening rules, respectively). The logic $\mathbf{FL_e}$ is known also as intuitionistic linear propositional logic without exponentials. Each substructural propositional logic over $\mathbf{FL_e}$ (i.e., each axiomatic extension of $\mathbf{FL_e}$) is called a *commutative* substructural propositional logic, as exchange rule is shown to be admissible in it.

The language of full Lambek propositional calculus **FL** consists of propositional constants $\top, \bot, 1, 0$, propositional variables and propositional connectives $\wedge, \vee, \cdot, \backslash$ and $/$. In algebraic terms, these constants are understood as the greatest element, the least element, the monoidal unit and the constant for defining the negation, respectively. Although it is known that $\top$ can be defined in **FL** by using $\bot$ and 0 as $\bot\backslash 0$ (and also as $0/\bot$), we will take it as a primitive symbol for the brevity's sake. In standard definition of substructural logics, only 1 and 0 are taken as propositional constants of the language while either $\top$ or $\bot$ is not, but the presence of constants $\top$ and $\bot$ in the language is essential for showing uniform interpolation property as explained later. Connectives $\wedge, \vee, \cdot, \backslash$ and $/$ are understood as conjunction, disjunction, fusion (or multiplicative conjunction) and two implications (called *left* and *right divisions*), respectively. Formulas are defined in the usual way. We introduce now three sequent calculi *without cut rule*. As a matter of fact, cut rule is admissible in any of them, as it is mentioned in Theorem 2.1 below. Except this the calculus **FL** is introduced in the standard way, while $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$ are introduced in a slightly different way (cf. Galatos *et al.*, 2007).

***2.1. Sequent calculus* FL.** A sequent of **FL** is any expression of the form either $\Gamma \Rightarrow \alpha$ or $\Gamma \Rightarrow$, where α is a formula and Γ is a (possibly empty) finite sequence of formulas separated by commas. Here $\Rightarrow$ is a metalogical symbol. As usual, when Γ and Σ are for finite sequences $\beta_1, \ldots, \beta_m$ and $\gamma_1, \ldots, \gamma_n$ of formulas, respectively, Γ, Σ (and Γ, α, Σ) expresses the sequence $\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n$ (and $\beta_1, \ldots, \beta_m, \alpha, \gamma_1, \ldots, \gamma_n$, respectively). The sequent calculus **FL** consists of initial sequents and rules given below, where φ may be possibly empty, p is any propositional variable, capital Greek letters Γ, Δ etc. denote (possibly empty) finite sequences of formulas.
Initial sequents:

(1) $p \Rightarrow p$
(2) $\Gamma \Rightarrow \top$
(3) $\Gamma, \bot, \Delta \Rightarrow \varphi$
(4) $\Rightarrow 1$
(5) $0 \Rightarrow$

https://doi.org/10.1017/S175502031400015X

Rules for logical connectives:

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, 1, \Delta \Rightarrow \varphi}\ (1w) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0w)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi}\ (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi}\ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow \wedge) \qquad \frac{\Gamma, \alpha, \Delta \Rightarrow \varphi \quad \Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \varphi}\ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \varphi}\ (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\Rightarrow \cdot)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Xi, \beta, \Delta \Rightarrow \varphi}{\Xi, \Gamma, \alpha \backslash \beta, \Delta \Rightarrow \varphi}\ (\backslash \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \backslash \beta}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Xi, \beta, \Delta \Rightarrow \varphi}{\Xi, \beta / \alpha, \Gamma, \Delta \Rightarrow \varphi}\ (/ \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta / \alpha}\ (\Rightarrow /)$$

***2.2. Sequent calculus* $\mathbf{FL_e}$.** Usually, $\mathbf{FL_e}$ is defined to be the sequent calculus obtained from **FL** by adding the following exchange rule (see, e.g., Galatos *et al.*, 2007);

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}\ (\text{ex} \Rightarrow)$$

To simply our discussions in the following, we will adopt a slightly different way of introducing $\mathbf{FL_e}$. That is, we define that a sequent of $\mathbf{FL_e}$ is any expression of the form either $\Gamma \Rightarrow \alpha$ or $\Gamma \Rightarrow$, where α is a formula and Γ is a (possibly empty) *multiset* of formulas (instead of finite sequence of formulas). It means that the order of formulas in Γ does not matter any more, and thus a sequent $\alpha, \alpha, \beta, \gamma \Rightarrow \delta$ can be identified with a sequent $\gamma, \alpha, \beta, \alpha \Rightarrow \delta$, for example.

Now, the calculus $\mathbf{FL_e}$ in the present paper is defined to be a system consisting of *the same initial sequents and rules*, but the antecedent (i.e., the left side of $\Rightarrow$) of each sequent is always understood as *a multiset of formulas*. Clearly, exchange rule becomes superfluous in our formulation, and in fact we can show that the provability of a sequent in our $\mathbf{FL_e}$ is equivalent under this identification to that in the usual $\mathbf{FL_e}$. In addition, formulas $\alpha\backslash\beta$ and β/α are mutually equivalent in $\mathbf{FL_e}$, i.e., both $\alpha\backslash\beta \Rightarrow \beta/\alpha$ and $\beta/\alpha \Rightarrow \alpha\backslash\beta$ are provable in $\mathbf{FL_e}$. Thus, in every substructural logic over $\mathbf{FL_e}$ we express $\alpha\backslash\beta$ as $\alpha \to \beta$ by taking the ordinary symbol $\to$ for implication. The negation $\neg\alpha$ of a formula α is also introduced in it as an abbreviation of $\alpha \to 0$.

***2.3. Sequent calculus* $\mathbf{FL_{ew}}$.** We will take the same approach as $\mathbf{FL_e}$ in defining the sequent calculus for $\mathbf{FL_{ew}}$, which is usually defined to be a calculus obtained from **FL** by

https://doi.org/10.1017/S175502031400015X

adding exchange rule mentioned above and also the following left- and right-weakening rules:

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}\ (\mathrm{w} \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}\ (\Rightarrow \mathrm{w})$$

It is easily seen that constants 1 and $\top$, and also constants 0 and $\bot$ are mutually equivalent in this $\mathbf{FL_{ew}}$. Therefore, constant 1 and 0 become superfluous and hence can be deleted in our language. Moreover, we can show that (a) the sequent $\Gamma \Rightarrow$ is provable in $\mathbf{FL_{ew}}$ if and only if the sequent $\Gamma \Rightarrow \bot$ in $\mathbf{FL_{ew}}$, and that (b) the rule (w $\Rightarrow$) can be deleted once we take the following initial sequent

(1') $\Gamma, p, \Delta \Rightarrow p$

instead of the initial sequent $p \Rightarrow p$. This is due to the fact that the rule (w $\Rightarrow$) is permutable with any other rule. Taking these facts into account, we get the following alternative sequent calculus, which we call $\mathbf{FL_{ew}}$ in the present paper.

The language of $\mathbf{FL_{ew}}$ consists of propositional constants $\top, \bot$, propositional variables and propositional connectives $\wedge, \vee, \cdot$ and $\rightarrow$. A sequent of $\mathbf{FL_{ew}}$ is any expression of the form either $\Gamma \Rightarrow \alpha$, where α is a formula and Γ is a (possibly empty) *multiset* of formulas. (Thus, we do not allow the sequent of the form $\Gamma \Rightarrow$ in $\mathbf{FL_{ew}}$ of the present paper. But this is only for the brevity's sake.) Initial sequents of $\mathbf{FL_{ew}}$ consists of the above (1'), in addition to initial sequents (2) and (3) of $\mathbf{FL_e}$, and rules of $\mathbf{FL_{ew}}$ are obtained from those for our $\mathbf{FL_e}$ by deleting both (1w) and (0w).

Though any of these three sequent calculi does not contain the following cut rule,

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \alpha, \Xi \Rightarrow \varphi}{\Sigma, \Gamma, \Xi \Rightarrow \varphi}\ (\mathrm{cut})$$

it can be shown that cut rule is *admissible* in each of them, i.e., adding cut rule does not change the set of all sequents provable in it. These facts are usually proved in the form of cut elimination theorem, were shown independently by several people. For instance, see Lambek (1958), Tamura (1974), Dardžaniá (1977), Idziak (1984), Ono & Komori (1985), Komori (1986), Došen (1988), and Ono (1990).

THEOREM 2.1. *Cut rule is admissible in each of* $\mathbf{FL}$, $\mathbf{FL_e}$ *and* $\mathbf{FL_{ew}}$.

We define the *weight* $w(\varphi)$ of a given formula φ inductively as follows: $w(p) = w(\top) = w(\bot) = w(1) = w(0) = 1$ where p is any propositional variable, and $w(\varphi \wedge \psi) = w(\varphi \vee \psi) = w(\varphi \cdot \psi) = w(\varphi \backslash \psi) = w(\psi / \varphi) = w(\varphi) + w(\psi) + 1$. Thus the weight $w(\varphi)$ denotes the total number of symbols in φ (except parentheses). The weight can be extended naturally to the weight $w(\Gamma)$ of each finite sequence or multiset Γ of formulas by the sum of the weight of its members. The weight of a given sequent $\Gamma \Rightarrow \alpha$ is defined to be $w(\Gamma, \alpha)$.

Take any one of calculus among $\mathbf{FL}$, $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$. Then it can be easily ascertained that in every rule the weight of (each of) its upper sequent(s) is strictly smaller than the weight of the lower sequent. (Note that the rule (ex $\Rightarrow$) is not used in our formulation of $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$.) From this we have the following corollary.

COROLLARY 2.2. *Substructural propositional logics* $\mathbf{FL}$, $\mathbf{FL_e}$ *and* $\mathbf{FL_{ew}}$ *are decidable.*

§3. Uniform interpolation for $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$. In this section, we will show uniform interpolation property of commutative substructural logics $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$. Uniform

https://doi.org/10.1017/S175502031400015X

interpolation property is a stronger form of *Craig interpolation property*. So we will first explain Craig interpolation property briefly in connection with uniform interpolation property. In the following, for each formula φ, $V(\varphi)$ denotes the set of all propositional variables appearing in φ. Hence, when $V(\varphi)$ is empty then φ must be a formula formed only from propositional constants. Similarly, for each finite sequence or multiset Γ of formulas, $V(\Gamma)$ we denote the set of all propositional variables appearing in some formula in Γ. A substructural propositional logic **L** over **FL** has *Craig interpolation property* if the following holds:

> for all formulas α and β, if $\alpha\backslash\beta$ is provable in **L** then there exists a formula γ such that both $\alpha\backslash\gamma$ and $\gamma\backslash\beta$ are provable in **L** and that $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

We can restate the above definition of Craig interpolation property by replacing all left divisions $\backslash$ in the above statement by right divisions $/$. This is due to the fact that $\alpha\backslash\beta$ is provable in **L** if and only if $\alpha \Rightarrow \beta$ is provable in **L** if and only if β/α is provable in **L**.

Maehara (1960) introduced a method of proving Craig interpolation property as a consequence of cut elimination. The method works well when it is applied to sequent calculi **FL**, $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$. In fact, we can prove the following lemmas, by using the induction of a given proof $\mathcal{P}$ of a sequent $\Sigma \Rightarrow \alpha$. From the proof we can get an interpolant θ constructively, which depends on the form of $\mathcal{P}$.

LEMMA 3.1. *If a sequent $\Sigma \Rightarrow \alpha$ is provable in* **FL** *then for any finite sequences Γ, Π and Δ such that Γ, Π, Δ is equal to Σ as sequences there exists a formula θ such that both $\Pi \Rightarrow \theta$ and $\Gamma, \theta, \Delta \Rightarrow \alpha$ are provable in* **FL** *and that $V(\theta) \subseteq V(\Pi) \cap V(\Gamma, \Delta, \alpha)$.*

For $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$, finite sequences must be replaced by multisets in the above lemma, and hence it will be rephrased as follows.

LEMMA 3.2. *If a sequent $\Sigma \Rightarrow \alpha$ is provable in $\mathbf{FL_e}$($\mathbf{FL_{ew}}$) then for any multisets Π and Δ such that the multiset union of Π and Δ is equal to Σ there exists a formula θ such that both $\Pi \Rightarrow \theta$ and $\theta, \Delta \Rightarrow \alpha$ are provable in $\mathbf{FL_e}$($\mathbf{FL_{ew}}$, respectively) and that $V(\theta) \subseteq V(\Pi) \cap V(\Delta, \alpha)$.*

Craig interpolation property follows immediately from these lemmas. See, e.g., Ono & Komori (1985), Ono (1990, 1998), and also Galatos *et al.* (2007).

COROLLARY 3.3. *Substructural propositional logics* **FL**, $\mathbf{FL_e}$ *and* $\mathbf{FL_{ew}}$ *have Craig interpolation property.*

A given logic **L** over **FL** has *uniform interpolation property* if and only if for each formula α and each set V_0 of propositional variables satisfying $V_0 \subseteq V(\alpha)$, there exists a formula $\alpha_*[V_0]$ such that

(1) $V(\alpha_*[V_0]) \subseteq V_0$,
(2) the formula $\alpha\backslash(\alpha_*[V_0])$ is provable in **L**,
(3) for any formula β such that $V(\alpha) \cap V(\beta) \subseteq V_0$, if $\alpha\backslash\beta$ is provable in **L** then $(\alpha_*[V_0])\backslash\beta$ is also provable in **L**

and also for each formula β and each set V_1 of propositional variables satisfying $V_1 \subseteq V(\beta)$, there exists a formula $\beta^*[V_1]$ such that

(4) $V(\beta^*[V_1]) \subseteq V_1$,
(5) the formula $(\beta^*[V_1])\backslash\beta$ is provable in **L**,

https://doi.org/10.1017/S175502031400015X

(6) for any formula α such that $V(\alpha) \cap V(\beta) \subseteq V_1$, if $\alpha\backslash\beta$ is provable in **L** then $\alpha\backslash(\beta^*[V_1])$ is also provable in **L**.

In such a case, the formula $\alpha_*[V_0]$ is called the *post-interpolant* of α with respect to V_0, and $\beta^*[V_1]$ the *pre-interpolant* of β with respect to V_1, respectively. They are determined uniquely up to logical equivalence in **L**. Suppose that a logic **L** has uniform interpolation property and that a formula $\alpha\backslash\beta$ is provable in **L**. Let $V = V(\alpha) \cap V(\beta)$. Then, $\alpha_*[V]$ and $\beta^*[V]$ are the strongest and the weakest, respectively, among interpolants of $\alpha\backslash\beta$.

We note here that the existence of constants $\bot$ and $\top$ is indispensable for showing uniform interpolation property of **FL** and $\mathbf{FL_e}$. For, otherwise there does not exist a post-interpolant of a given propositional variable p with respect to the empty set $\emptyset$ of variables. In fact, if γ were such a post-interpolant then it should be a formula constructed only by constants 0 and 1 and moreover the formula $p\backslash\gamma$ should be provable in **FL** (or $\mathbf{FL_e}$). But this is shown to be impossible.

To confirm uniform interpolation property it is necessary to show the existence of post- and pre-interpolants of each formula α with respect to each subset V_0 of $V(\alpha)$, due to the above definition. But, this can be simplified as follows.

LEMMA 3.4. *Suppose that for each formula α and each $p \in V(\alpha)$ there exists a post-interpolant of α in* **L** *with respect to $V(\alpha)\backslash\{p\}$. Then a post-interpolant in* **L** *of α with respect to V_0 exists for each $V_0 \subseteq V(\alpha)$. Similarly, the similar statement holds for pre-interpolants.*

Proof. This is shown by using the induction on the number n of variables in $V(\alpha)\backslash V$. When $n = 0$, i.e., when $V_0 = V(\alpha)$, it is clear that α itself is the post-interpolant of α with respect to $V(\alpha)$, and hence it suffices to take α for $\alpha_*[V(\alpha)]$. This is also trivial by our assumption when $n = 1$. Suppose that this holds for $n \leq k$, i.e., a post-interpolant of each formula γ with respect to each subset V of $V(\gamma)$ as long as the number of variables in $V(\gamma)\backslash V$ is less than $k + 1$. We assume now that for a given formula α, V_0 is a subset of $V(\alpha)$ such that the number of variables in $V(\alpha)\backslash V_0$ is $k + 1$. Take an arbitrary variable $p \in V(\alpha)\backslash V_0$. Then by our assumption, there exists a post-interpolant $\alpha_*[V(\alpha)\backslash\{p\}]$ of α with respect to $V(\alpha)\backslash\{p\}$. Let us denote it as δ. Then it satisfies;

(a) $V(\delta) \subseteq V(\alpha)\backslash\{p\}$,

(b) the formula $\alpha\backslash\delta$ is provable in **L**,

(c) for any formula β such that $V(\alpha) \cap V(\beta) \subseteq V(\alpha)\backslash\{p\}$, or equivalently $p \notin V(\beta)$, if $\alpha\backslash\beta$ is provable in **L** then $\delta\backslash\beta$ is also provable in **L**.

Define $V_0{}' = V_0 \cap V(\delta)$. Since $V(\delta)\backslash V_0{}' \subseteq \{V(\alpha)\backslash\{p\}\}\backslash V_0$, the number of variables in $V(\delta)\backslash V_0{}'$ is less than $k + 1$. By using the hypothesis of induction, there exists a formula $\delta_*[V_0{}']$ such that

(d) $V(\delta_*[V_0{}']) \subseteq V_0{}' \subseteq V_0$,

(e) the formula $\delta\backslash\delta_*[V_0{}']$ is provable in **L**,

(f) for any formula β such that $V(\delta) \cap V(\beta) \subseteq V_0{}'$, if $\delta\backslash\beta$ is provable in **L** then $\delta_*[V_0{}']\backslash\beta$ is also provable in **L**.

By (b) and (e), $\alpha\backslash\delta_*[V_0{}']$ is provable in **L**. Next suppose that $V(\alpha) \cap V(\beta) \subseteq V_0$ and $\alpha\backslash\beta$ is provable in **L**. Clearly, $p \notin V(\beta)$. Hence $\delta\backslash\beta$ is provable by (c). From $V(\delta) \cap V(\beta) \subseteq V(\alpha) \cap V(\beta) \subseteq V_0$ and $V(\delta) \cap V(\beta) \subseteq V(\delta)$, it follows that $V(\delta) \cap V(\beta) \subseteq V_0{}'$. Thus, $\delta_*[V_0{}']\backslash\beta$ is provable by (f). This means that the formula $\delta_*[V_0{}']$ is a post-interpolant of α with respect to V_0. □

https://doi.org/10.1017/S175502031400015X

Table 1. *Definition of $E_p(\Gamma)$ for* $\mathbf{FL_e}$

Γ matches	$\mathcal{E}_p(\Gamma)$ contains
$\emptyset$	1
p	$\top$
q	q
$\Gamma', \varphi \wedge \psi$	$E_p(\Gamma', \varphi) \wedge E_p(\Gamma', \psi)$
$\Gamma', \varphi \vee \psi$	$E_p(\Gamma', \varphi) \vee E_p(\Gamma', \psi)$
$\Gamma', \varphi \cdot \psi$	$E_p(\Gamma', \varphi, \psi)$
$\Gamma', \varphi \to \psi$	$\bigwedge_{\Gamma_1, \Gamma_2 = \Gamma'} [A_p(\Gamma_1; \varphi) \to E_p(\Gamma_2, \psi)]$
Γ	$\bigwedge_{\Gamma_1, \Gamma_2 = \Gamma} E_p(\Gamma_1) \cdot E_p(\Gamma_2)$
otherwise	$\top$

In the following, we show the uniform interpolation property of $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$ by using the idea of A. Pitts (Pitts, 1992). First, we introduce formulas $E_p(\Gamma)$ and $A_p(\Gamma; \alpha)$ for $\mathbf{FL_e}$ in Tables 1 and 2, where p is any propositional variable, α a formula or may be empty, and Γ a (possibly empty) multiset of formulas. It will be shown in Theorem 3.5 that formulas $E_p(\alpha)$ and $A_p(\emptyset; \alpha)$ are explicit representations of $\alpha_*[V(\alpha)\backslash\{p\}]$ and $\alpha^*[V(\alpha)\backslash\{p\}]$, respectively. In this way, we can give a syntactic theoretic proof of the uniform interpolation property of $\mathbf{FL_e}$ (Corollary 3.6).

We assume in all tables in the present paper (i.e., from Tables 1 to 9) that *p is a fixed propositional variable (predicate symbol) under consideration, q is either any propositional variable (predicate symbol) other than p or a constant, and r is either any propositional variable (predicate symbol, respectively, in Table 9) or a constant.* In this and the next sections, we will use $\to$ as the symbol for implication, instead of $\backslash$, as all logics discussed in them are commutative. We require that in the second row to the last in Table 1, *both Γ_1 and Γ_2 are nonempty.*

Both formulas $E_p(\Gamma)$ and $A_p(\Gamma; \alpha)$ are defined simultaneously by induction on the weight of Γ and Γ, α, respectively. More precisely, in each induction step, sets of formulas $\mathcal{E}_p(\Gamma)$ and $\mathcal{A}_p(\Gamma; \alpha)$ are defined first. Since both of them will be shown to be finite sets of formulas, we define next the formulas $E_p(\Gamma)$ and $A_p(\Gamma; \alpha)$, by taking the conjunction of all formulas in $\mathcal{E}_p(\Gamma)$ and by taking the disjunction of all formulas in $\mathcal{A}_p(\Gamma; \alpha)$, respectively.

We remark here that both $\mathcal{E}_p(\Gamma)$ and $\mathcal{A}_p(\Gamma; \alpha)$ may contain more than one formula, and therefore the statement "a set $\mathcal{A}_p(\Gamma; \alpha)$ contains a formula β in Table 2" does not imply always that $A_p(\Gamma; \alpha)$ is equal to β. For example, $\mathcal{A}_p(\alpha \vee \beta; \gamma \vee \delta)$ contains $A_p(\alpha \vee \beta; \gamma) \vee A_p(\alpha \vee \beta; \delta)$ but $A_p(\alpha \vee \beta; \gamma \vee \delta)$ may not be equal to it, as $\mathcal{A}_p(\alpha \vee \beta; \gamma \vee \delta)$ contains also $A_p(\alpha; \gamma \vee \delta) \wedge A_p(\beta; \gamma \vee \delta)$. Our goal of this section is to show the following theorem.

Table 2. *Definition of* $A_p(\Gamma;\alpha)$ *for* $\mathbf{FL_e}$

$\Gamma;\alpha$ matches	$\mathcal{A}_p(\Gamma;\alpha)$ contains
$\emptyset;\emptyset$	0
$0;\emptyset$	1
$\emptyset;q$	q
$r;r$	1
$\Gamma;0$	$A_p(\Gamma;\emptyset)$
$\Gamma;\top$	$\top$
$\Gamma',\varphi\wedge\psi;\alpha$	$A_p(\Gamma',\varphi;\alpha)\vee A_p(\Gamma',\psi;\alpha)$
$\Gamma',\varphi\vee\psi;\alpha$	$A_p(\Gamma',\varphi;\alpha)\wedge A_p(\Gamma',\psi;\alpha)$
$\Gamma',\varphi\cdot\psi;\alpha$	$A_p(\Gamma',\varphi,\psi;\alpha)$
$\Gamma',\varphi\to\psi;\alpha$	$\bigvee_{\Gamma_1,\Gamma_2=\Gamma'} A_p(\Gamma_1;\varphi)\cdot A_p(\Gamma_2,\psi;\alpha)$
$\Gamma',1;\alpha$	$A_p(\Gamma';\alpha)$
$\Gamma',\bot;\alpha$	$\top$
$\Gamma;\alpha$	$\bigvee_{\Gamma_1,\Gamma_2=\Gamma \text{and} \Gamma_1\neq\emptyset}[E_p(\Gamma_1)\to A_p(\Gamma_2;\alpha)]$
$\Gamma;\varphi\wedge\psi$	$A_p(\Gamma;\varphi)\wedge A_p(\Gamma;\psi)$
$\Gamma;\varphi\vee\psi$	$A_p(\Gamma;\varphi)\vee A_p(\Gamma;\psi)$
$\Gamma;\varphi\cdot\psi$	$\bigvee_{\Gamma_1,\Gamma_2=\Gamma} A_p(\Gamma_1;\varphi)\cdot A_p(\Gamma_2;\psi)$
$\Gamma;\varphi\to\psi$	$A_p(\Gamma,\varphi;\psi)$
otherwise	$\bot$

THEOREM 3.5. *Let* Γ *be a multiset of formulas and* α *be a single formula or empty. For every propositional variable* p *there exist formulas* $E_p(\Gamma)$ *and* $A_p(\Gamma;\alpha)$ *such that*

(1) (a) $V(E_p(\Gamma))\subseteq V(\Gamma)\backslash\{p\}$,
(b) $V(A_p(\Gamma;\alpha))\subseteq V(\Gamma,\alpha)\backslash\{p\}$.

(2) (a) $\Gamma\Rightarrow E_p(\Gamma)$ *is provable in* $\mathbf{FL_e}$,
(b) $\Gamma, A_p(\Gamma;\alpha)\Rightarrow\alpha$ *is provable in* $\mathbf{FL_e}$.

(3) *Let* Π *be any multiset of formulas not containing* p. *If* $\Pi,\Gamma\Rightarrow\alpha$ *is provable in* $\mathbf{FL_e}$, *then*

https://doi.org/10.1017/S175502031400015X

(a) $\Pi, E_p(\Gamma) \Rightarrow \alpha$ *is provable in* $\mathbf{FL_e}$*, when* $p \notin V(\alpha)$*,*

(b) $\Pi \Rightarrow A_p(\Gamma; \alpha)$ *is provable in* $\mathbf{FL_e}$.

Proof. We will devote the rest of this section to the proof of Theorem 3.5. The first statements 1(a) and 1(b) are trivial. The second statements 2(a) and 2(b) are confirmed by using simultaneous induction on the weight. More precisely, in the induction step, we need to show that (1) when $w(\Gamma) = w_0$, a sequent $\Gamma \Rightarrow E_p(\Gamma)$ is provable in $\mathbf{FL_e}$ under the assumption that every sequent of the form $\Gamma' \Rightarrow E_p(\Gamma')$ and $\Sigma', A_p(\Sigma'; \beta) \Rightarrow \beta$ are provable in $\mathbf{FL_e}$ whenever $w(\Gamma') = w(\Sigma', \beta) < w_0$, and (2) when $w(\Sigma, \alpha) = w_0$, a sequent $\Sigma, A_p(\Sigma; \alpha) \Rightarrow \alpha$ are provable in $\mathbf{FL_e}$ under the assumption that every sequent of the form $\Gamma' \Rightarrow E_p(\Gamma')$ and $\Sigma', A_p(\Sigma'; \beta) \Rightarrow \beta$ are provable in $\mathbf{FL_e}$ whenever $w(\Gamma') \leq w_0$ and $w(\Sigma', \beta) < w_0$.

It is not hard to show the second statements 2(a) and 2(b) for each of the initial steps. What actually we need to show is that $\Gamma \Rightarrow \beta$ is provable in $\mathbf{FL_e}$ for each $\beta \in \mathcal{E}_p(\Gamma)$ and that $\Gamma, \delta \Rightarrow \alpha$ is provable in $\mathbf{FL_e}$ for each $\delta \in \mathcal{A}_p(\Gamma; \alpha)$. (In the rest of this section, sometimes “provable in $\mathbf{FL_e}$" is expressed simply as “provable.")

We consider here the cases for Γ is of the form $\Gamma', \varphi \to \psi$. Take arbitrary multisets Γ_1 and Γ_2 such that their multiset union Γ_1, Γ_2 is equal to Γ'. Since either of weights $w(\Gamma_1, \varphi)$ and $w(\Gamma_2, \psi)$ is smaller than $w(\Gamma', \varphi \to \psi)$, sequents $\Gamma_1, A_p(\Gamma_1; \varphi) \Rightarrow \varphi$ and $\Gamma_2, \psi \Rightarrow E_p(\Gamma_2, \psi)$ are provable by using the hypothesis of induction. Using ($\to\Rightarrow$), we get $\Gamma', \varphi \to \psi, A_p(\Gamma_1; \varphi) \Rightarrow E_p(\Gamma_2, \psi)$ and hence $\Gamma', \varphi \to \psi \Rightarrow A_p(\Gamma_1; \varphi) \to E_p(\Gamma_2, \psi)$ is provable. Thus, we have that $\Gamma', \varphi \to \psi \Rightarrow \bigwedge_{\Gamma_1, \Gamma_2 = \Gamma'}[A_p(\Gamma_1; \varphi) \to E_p(\Gamma_2, \psi)]$ is also provable. This is what we need to show for the case where $\Gamma = \Gamma'$, $\varphi \to \psi$.

For the statement 2(b), take Γ_1 and Γ_2 as before. By using the hypothesis of induction, both sequents $\Gamma_1, A_p(\Gamma_1; \varphi) \Rightarrow \varphi$ and $\Gamma_2, \psi, A_p(\Gamma_2, \psi; \alpha) \Rightarrow \alpha$ are provable. Hence, $\Gamma', \varphi \to \psi, A_p(\Gamma_1; \varphi) \cdot A_p(\Gamma_2, \psi; \alpha) \Rightarrow \alpha$ is provable. Since this holds for arbitrary Γ_1 and Γ_2 such that $\Gamma_1, \Gamma_2 = \Gamma'$, we have the required result.

We show next the third statements 3(a) and 3(b). Again we use the induction on the weight of $\Pi, \Gamma \Rightarrow \alpha$. (Note that the proof is not carried out by using the induction of the *length* of a given proof, different from the proof of Craig interpolation property by Maehara's method. As mentioned just above Lemma 3, an interpolant given by Maehara's method is determined by a given proof. On the other hand, to guarantee the uniformity of pre- and post-interpolants, we must consider all (but finitely many) proofs of a given sequent $\Pi, \Gamma \Rightarrow \alpha$.) This time, it suffices to show that $\Pi, \beta \Rightarrow \alpha$ is provable for some $\beta \in \mathcal{E}_p(\Gamma)$ when $p \notin V(\{\alpha\})$, and $\Pi \Rightarrow \delta$ is provable for some $\delta \in \mathcal{A}_p(\Gamma; \alpha)$.

Every initial step consists of cases where a given sequent $\Pi, \Gamma \Rightarrow \alpha$ is an initial sequent. Let us consider the case where $\Pi, \Gamma \Rightarrow \alpha$ is of the form $r \Rightarrow r$ where r is a propositional variable. Obviously, α is r. Now suppose that r is different from p. If Π is r then Γ is empty. Since $\mathcal{E}_p(\Gamma)$ contains 1 and $r, 1 \Rightarrow r$ is provable, 3(a) holds. Note that $\mathcal{A}_p(\emptyset; r)$ contains r. Thus 3(b) holds. If Π is empty, Γ is r. Then 3(a) holds as $\mathcal{E}_p(r)$ contains r and $r \Rightarrow r$ is provable. Also 3(b) holds, since $\mathcal{A}_p(r; r)$ contains 1. Suppose next that r is p. Then, Π must be empty, and Γ is equal to p. It is enough to show 3(b). But it can be shown similarly as the last case of the previous case.

We consider next the induction step. Suppose that the sequent $\Pi, \Gamma \Rightarrow \alpha$ is provable, where Π is a multiset of formulas not containing p. We discuss here only the cases where it is obtained by applying either ($\to\Rightarrow$) or ($\Rightarrow\to$). In the first case, let $\Pi_1, \Gamma_1 \Rightarrow \varphi$ and $\Pi_2, \Gamma_2, \psi \Rightarrow \alpha$ be the upper sequents for some Π_1, Π_2, Γ_1 and Γ_2, where Π_1 and Π_2 are sub multisets of Π, and Γ_1 and Γ_2 are sub multisets of Γ, respectively. Note that the weight of either of upper sequents is smaller than that of $\Pi, \Gamma \Rightarrow \alpha$. Thus, we can apply the hypothesis of induction to them. We consider the following two cases depending on whether $\varphi \rightarrow \psi \in \Pi$ or not.

Case 1. Assume that $\varphi \rightarrow \psi \in \Pi$. Hence, $\Pi = \Pi_1, \Pi_2, \varphi \rightarrow \psi$ and $\Gamma = \Gamma_1, \Gamma_2$. Our assumption implies that neither φ nor ψ contains p. We show first that the statement 3(a) holds, assuming that $p \notin V(\alpha)$. Let us suppose moreover that both Γ_1 and Γ_2 are nonempty. By the hypothesis of induction, sequents $\Pi_1, E_p(\Gamma_1) \Rightarrow \varphi$ and $\Pi_2, \psi, E_p(\Gamma_2) \Rightarrow \alpha$ are provable. Applying first $(\rightarrow\Rightarrow)$ and then $(\cdot \Rightarrow)$ to them, we get $\Pi_1, \Pi_2, \varphi \rightarrow \psi, E_p(\Gamma_1) \cdot E_p(\Gamma_2) \Rightarrow \alpha$. Thus, $\Pi_1, \Pi_2, \varphi \rightarrow \psi, \bigwedge_{\Gamma_1, \Gamma_2 = \Gamma \text{and } \Gamma_i \neq \emptyset} E_p(\Gamma_1) \cdot E_p(\Gamma_2) \Rightarrow \alpha$. When $\Gamma_1 = \emptyset$ and $\Gamma_2 = \Gamma$, both $\Pi_1 \Rightarrow \varphi$ and $\Pi_2, \psi, E_p(\Gamma) \Rightarrow \alpha$ are provable, and hence $\Pi_1, \Pi_2, \varphi \rightarrow \psi, E_p(\Gamma) \Rightarrow \alpha$ is also provable. Similarly, this holds also when $\Gamma_1 = \Gamma$ and $\Gamma_2 = \emptyset$.

Next we show the statement 3(b). First we assume that Γ_1 is nonempty. By the hypothesis of induction, sequents $\Pi_1, E_p(\Gamma_1) \Rightarrow \varphi$ and $\Pi_2, \psi \Rightarrow A_p(\Gamma_2; \alpha)$ are provable. By using $(\rightarrow\Rightarrow)$ and then $(\Rightarrow\rightarrow)$, we have $\Pi_1, \Pi_2, \varphi \rightarrow \psi \Rightarrow E_p(\Gamma_1) \rightarrow A_p(\Gamma_2; \alpha)$. From this and the definition of $A_p(\Gamma; \alpha)$, the provability of $\Pi \Rightarrow A_p(\Gamma; \alpha)$ follows. When Γ_1 is empty, $\Gamma_2 = \Gamma$. In this case, both $\Pi_1, \Rightarrow \varphi$ and $\Pi_2, \psi \Rightarrow A_p(\Gamma; \alpha)$ are provable. Hence, $\Pi_1, \Pi_2, \varphi \rightarrow \psi \Rightarrow A_p(\Gamma; \alpha)$ is provable.

Case 2. Suppose otherwise. Then $\Pi = \Pi_1, \Pi_2$ and $\Gamma = \Gamma_1, \Gamma_2, \varphi \rightarrow \psi$. First we assume that $p \notin V(\alpha)$. By the hypothesis of induction, both $\Pi_1 \Rightarrow A_p(\Gamma_1; \varphi)$ and $\Pi_2, E_p(\Gamma_2, \psi) \Rightarrow \alpha$ are provable. Applying $(\rightarrow\Rightarrow)$ to them, $\Pi, A_p(\Gamma_1; \varphi) \rightarrow E_p(\Gamma_2, \psi) \Rightarrow \alpha$ is provable. By using $(\wedge \Rightarrow)$, $\Pi, \bigwedge_{\Gamma_1, \Gamma_2 = \Gamma} [A_p(\Gamma_1; \varphi) \rightarrow E_p(\Gamma_2, \psi)] \Rightarrow \alpha$ is shown to be provable, from which the provability of $\Pi, E_p(\Gamma) \Rightarrow \alpha$ follows. For the statement 3(b), both $\Pi_1 \Rightarrow A_p(\Gamma_1; \varphi)$ and $\Pi_2 \Rightarrow A_p(\Gamma_2, \psi; \alpha)$ are provable, using the hypothesis of induction. Thus, $\Pi \Rightarrow A_p(\Gamma_1; \varphi) \cdot A_p(\Gamma_2, \psi; \alpha)$ is provable. Thus, $\Pi \Rightarrow A_p(\Gamma; \alpha)$ is provable.

We discuss next the case where the last rule applied is $(\Rightarrow\rightarrow)$, i.e., the sequent under consideration is of the form $\Pi, \Gamma \Rightarrow \varphi \rightarrow \psi$. Thus the upper sequent must be $\Pi, \Gamma, \varphi \Rightarrow \psi$. When $p \notin V(\varphi \rightarrow \psi)$, obviously $p \notin V(\psi)$. Hence by using the hypothesis of induction, $\Pi, E_p(\Gamma), \varphi \Rightarrow \psi$ is provable, from which the provability of $\Pi, E_p(\Gamma) \Rightarrow \varphi \rightarrow \psi$ follows. For 3(b), $\Pi \Rightarrow A_p(\Gamma, \varphi; \psi)$ is provable by the hypothesis of induction. Hence the provability of $\Pi \Rightarrow A_p(\Gamma; \varphi \rightarrow \psi)$ follows from the definition of $A_p(\Gamma; \varphi \rightarrow \psi)$. □

COROLLARY 3.6. *The uniform interpolation property holds for* **FLe**.

Similarly to **FLe**, we can show the uniform interpolation for the logic **FLew**. To get a similar result to Theorem 3.5 for **FLew**, it is necessary to modify Tables 1 and 2 in the following way. In the definition of $E_p(\Gamma)$, we replace the first 3 lines of the original table by the following, and also in the definition of $A_p(\Gamma; \alpha)$, among the first 5 lines delete the first, the second and the fifth lines and then replace the fourth lines by the following, while keeping the third line as it stands. (As a matter of fact, the sixth line in the previous table becomes redundant in the new table.)

We can show a result for **FLew** which corresponds to Theorem 3.5, in the same way as our proof of Theorem 3.5. But some additional calculations using induction will be necessary in order to confirm that the modified $E_p(\Gamma', r)$ for any variable r and $A_p(\Gamma; q)$ for any variable q except p satisfy the condition 3(a) and 3(b). respectively. As a consequence, we have the following.

https://doi.org/10.1017/S175502031400015X

Table 3. *Modified* $E_p(\Gamma)$ *and* $A_p(\Gamma; \alpha)$ *for* $\mathbf{FL_{ew}}$

Γ matches	$\mathcal{E}_p(\Gamma)$ contains
$\emptyset$	$\top$
Γ', p	$\top$
Γ', q	q
$\Gamma; \alpha$ matches	$\mathcal{A}_p(\Gamma; \alpha)$ contains
$\Gamma; q$	q
$\Gamma', r; r$	$\top$

THEOREM 3.7. *The uniform interpolation property holds for* $\mathbf{FL_{ew}}$.

Example 1. Let us consider the following sequent: $p, s, \neg(p \wedge q) \Rightarrow q \rightarrow (r \vee s)$, which is provable in $\mathbf{FL_{ew}}$. Variables which appear common in the left side and the right side of the sequent are q and s. It can be easily observed that $\neg q$ and s are interpolants in $\mathbf{FL_{ew}}$. Hence, $\neg q \wedge s$ and $\neg q \vee s$ are also interpolants. Moreover, $q \rightarrow s$ is an interpolant. Consider the set $\mathcal{E}_p(p, s, \neg(p \wedge q))$, using Tables for $\mathbf{FL_{ew}}$. As $\neg(p \wedge q)$ is an abbreviation of $(p \wedge q) \rightarrow \bot$, it contains all formulas of the form $A_p(\Gamma_1; p \wedge q) \rightarrow E_p(\Gamma_2, \bot)$, which is equal to $\neg A_p(\Gamma_1; p \wedge q)$, where the multiset union of Γ_1 and Γ_2 must be equal to the multiset $\{p, s\}$, and contains also all formulas of the form $E_p(\Delta_1) \cdot E_p(\Delta_2)$ where both Δ_1 and Δ_2 are nonempty multisets such that their multiset union is equal to the multiset $\{p, s, \neg(p \wedge q)\}$. Take the conjunction of each of them. Then, the first one is $\neg q$ and the second is s. Thus, $E_p(p, s, \neg(p \wedge q))$ is $\neg q \wedge s$. Similarly, we can show that $A_r(\emptyset; q \rightarrow (r \vee s))$ is equal to $q \rightarrow s$.

§4. Involutive logics $\mathbf{InFL_e}$ and $\mathbf{InFL_{ew}}$.

The approach to uniform interpolation property in the previous section can be applied also to substructural logics $\mathbf{InFL_e}$ and $\mathbf{InFL_{ew}}$ which are involutive extensions (i.e., extensions by adding *the law of double negation* $\neg\neg\varphi \rightarrow \varphi$) of $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$, respectively. They are sometimes called linear logic (denoted by **MALL**) and affine logic (without exponentials), respectively.

As usual, we will take multi-succedent sequent calculi in order to formalize them. Hence, sequents of calculi $\mathbf{InFL_e}$ and $\mathbf{InFL_{ew}}$ are of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are (possibly empty) multisets. Roughly speaking, rules of $\mathbf{InFL_e}$ and $\mathbf{InFL_{ew}}$ are obtained from those of $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$ given in the previous section, by replacing any formula φ by a multiset Φ, and any sequent of the form $\Theta \Rightarrow \gamma$ such that γ is different from φ by $\Theta \Rightarrow \gamma, \Lambda$ with a multiset Λ. An important feature of these calculi lies in their duality. For this purpose, it is convenient to use a unary connective $\neg$ for negation and a binary connective $+$, the dual of fusion as primitive logical connectives, while the implication $\rightarrow$ is regarded as a symbol such that $\alpha \rightarrow \beta$ is defined by $\neg\alpha + \beta$. It can be shown that $\alpha + \beta$ is equivalent to $\neg(\neg\alpha \cdot \neg\beta)$ in these calculi. Here is a precise definition of the sequent calculus $\mathbf{InFL_e}$.

https://doi.org/10.1017/S175502031400015X

Initial sequents:

(1) $p \Rightarrow p$ for any propositional variable p,
(2) $\Gamma \Rightarrow \top, \Delta$
(3) $\Gamma, \bot \Rightarrow \Delta$
(4) $\Rightarrow 1$
(5) $0 \Rightarrow$

Rules for logical connectives:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, 1 \Rightarrow \Delta}\ (1 \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow 0, \Delta}\ (\Rightarrow 0)$$

$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Gamma, \neg\alpha \Rightarrow \Delta}\ (\neg \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \neg\alpha, \Delta}\ (\Rightarrow \neg)$$

$$\frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow \Delta}\ (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow \Delta}\ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha, \Delta \quad \Gamma \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \wedge \beta, \Delta}\ (\Rightarrow \wedge) \qquad \frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta}\ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Gamma \Rightarrow \alpha \vee \beta, \Delta}\ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \vee \beta, \Delta}\ (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \alpha, \beta \Rightarrow \Delta}{\Gamma, \alpha \cdot \beta \Rightarrow \Delta}\ (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha, \Delta \quad \Sigma \Rightarrow \beta, \Lambda}{\Gamma, \Sigma \Rightarrow \alpha \cdot \beta, \Delta, \Lambda}\ (\Rightarrow \cdot)$$

$$\frac{\Gamma, \alpha \Rightarrow \Delta \quad \Sigma, \beta \Rightarrow \Lambda}{\Gamma, \Sigma, \alpha + \beta \Rightarrow \Delta, \Lambda}\ (+ \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha, \beta, \Delta}{\Gamma \Rightarrow \alpha + \beta, \Delta}\ (\Rightarrow +)$$

From these rules, one can see the duality between $\top$ and $\bot$, 1 and 0, $\wedge$ and $\vee$, and $\cdot$ and $+$. Therefore, in each of these pairs one is defined by the other using the negation $\neg$. The following theorem holds (see, e.g., Girard, 1987; Troelstra, 1992).

THEOREM 4.1. *Cut rule is admissible in* $\mathbf{InFL_e}$.

By the definition of the uniform interpolation property, it will be necessary to show the existence of both the post-interpolation and the pre-interpolation for each formula γ and each subset V of $V(\gamma)$. But as long as the law of double negation holds in a logic $\mathbf{L}$, the existence of one of them implies the existence of the other, as shown below (cf. Bílková, 2007).

LEMMA 4.2. *When the law of double negation holds in a logic* $\mathbf{L}$*, the pre-interpolant exists always if and only if the post-interpolant exists always.*

Proof. Though this lemma holds also for noncommutative logics, we give a proof of this lemma when $\mathbf{L}$ is commutative. (See Galatos *et al.*, 2007 for the law of double negation in noncommutative logics.) Suppose that the pre-interpolant exists always for each formula γ and each subset V of $V(\gamma)$. Let α be an arbitrary formula and V_0 be an arbitrary subset of

https://doi.org/10.1017/S175502031400015X

$V(\alpha)$. We show that the formula $\neg(\neg\alpha)^*[V_0]$ is the post-interpolant with respect to α and V_0. First note that $V(\neg(\neg\alpha)^*[V_0]) = V((\neg\alpha)^*[V_0]) \subseteq V_0$. Next, since $(\neg\alpha)^*[V_0] \to \neg\alpha$ is provable in **L** as $(\neg\alpha)^*[V_0]$ is the pre-interpolant, $\alpha \to \neg(\neg\alpha)^*[V_0]$ must be provable in it. Lastly, let β be any formula such that $V(\alpha) \cap V(\beta) \subseteq V_0$ and that $\alpha \to \beta$ is provable in **L**. Then $\neg\beta \to \neg\alpha$ is provable, and hence $\neg\beta \to (\neg\alpha)^*[V_0]$ is provable, by the definition of the pre-interpolant. This implies that $\neg(\neg\alpha)^*[V_0] \to \beta$ is also provable in **L**, by using the law of double negation. Thus, $\neg(\neg\alpha)^*[V_0]$ is the post-interpolant. The converse implication can be shown similarly. □

Because of the above lemma, it is enough to introduce the notion of $A_p(\Gamma; \Delta)$ for arbitrary multisets Γ and Δ, which is an extension of $A_p(\Gamma; \alpha)$ introduced in the previous section. *In the following Table 4, we assume that* $\Gamma_i = \Delta_i = \emptyset$ *is not allowed for* $i \in \{1, 2\}$ *in the definition of* $\mathcal{A}_p(\Gamma_1, \Gamma_2; \Delta_1, \Delta_2)$.

Corresponding to Theorem 3.5, the following two results can be shown similarly to it and its corollary. (Compare this with Theorem 5.1 in Bílková, 2007.)

THEOREM 4.3. *Let* Γ *and* Δ *be arbitrary multisets of formulas. For every propositional variable* p *there exist a formula* $A_p(\Gamma; \Delta)$ *such that:*

(1) $V(A_p(\Gamma; \Delta)) \subseteq V(\Gamma, \Delta)\backslash\{p\}$.
(2) $\Gamma, A_p(\Gamma; \Delta) \Rightarrow \Delta$ *is provable in* $\mathbf{InFL_e}$.
(3) *Let* Π *and* Λ *be arbitrary multisets of formulas not containing* p. *If* $\Pi, \Gamma \Rightarrow \Delta, \Lambda$ *is provable in* $\mathbf{InFL_e}$, *then* $\Pi \Rightarrow A_p(\Gamma; \Delta), \Lambda$ *is provable in* $\mathbf{InFL_e}$.

COROLLARY 4.4. *The uniform interpolation property holds for* $\mathbf{InFL_e}$.

Our calculus $\mathbf{InFL_{ew}}$ is obtained from $\mathbf{InFL_e}$ by deleting all of its initial sequents and rules related to constants 1 and 0, and then adding every sequent of the form $\Gamma, \alpha \Rightarrow \alpha, \Delta$ as new initial sequents. In the same way as $\mathbf{FL_{ew}}$, we can show a result on $\mathbf{InFL_{ew}}$ which corresponds to Theorem 4.3 by modifying the 1st, the 3rd and the 4th lines of Table 4 as shown in Table 5. Note that the admissibility of cut rule in $\mathbf{InFL_{ew}}$ is proved in Grišin (1982). Thus we have the following.

THEOREM 4.5. *The uniform interpolation property holds for* $\mathbf{InFL_{ew}}$.

§5. Full Lambek calculus FL.

One may suppose that our arguments will work also for noncommutative substructural logics like full Lambek calculus **FL** by simply replacing finite sequences of formulas in sequents by multisets of formulas. But, if we try to define E_p and A_p for **FL** we will face a certain difficulties caused by the noncommutativity of formulas in a given sequent. Let us consider the following rule for left division;

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow \backslash).$$

As one of rules of **FL**, it means that we can derive $\Gamma \Rightarrow \alpha\backslash\beta$ only when α is *the left-most formula* in the left side of the upper sequent. On the other hand, when we understand the left side of each sequent to be a multiset of formulas, we can derive $\Gamma \Rightarrow \alpha\backslash\beta$ whenever α is a member of the left side of the upper sequent. This difference will affect the definition of E_p and A_p for **FL**. Let us look at our proof of 2(a) of Theorem 3.5 on $\mathbf{FL_e}$ for the case when Γ is of the form $\Gamma', \varphi \to \psi$, and compare this with the corresponding case for **FL**. In the former case, we could show that $\Gamma', \varphi \to \psi \Rightarrow \bigwedge_{\Gamma_1,\Gamma_2=\Gamma'}[A_p(\Gamma_1; \varphi) \to E_p(\Gamma_2, \psi)]$ is provable, where Γ_1 and Γ_2 are any multisets of formulas such that their multiset union

https://doi.org/10.1017/S175502031400015X

Table 4. *Definition of* $A_p(\Gamma; \Delta)$ *for* $\mathbf{InFL_e}$

$\Gamma; \Delta$ matches	$\mathcal{A}_p(\Gamma; \Delta)$ contains
$r; r$	1
$\emptyset; \emptyset$	0
$q; \emptyset$	$\neg q$
$\emptyset; q$	q
$\Gamma', 1; \Delta$	$A_p(\Gamma'; \Delta)$
$\Gamma; 0, \Delta'$	$A_p(\Gamma; \Delta')$
$\Gamma; \top, \Delta'$	$\top$
$\Gamma', \bot; \Delta$	$\top$
$\Gamma; \varphi \wedge \psi, \Delta'$	$A_p(\Gamma; \varphi, \Delta') \wedge A_p(\Gamma; \psi, \Delta')$
$\Gamma; \varphi \vee \psi, \Delta'$	$A_p(\Gamma; \varphi, \Delta') \vee A_p(\Gamma; \psi, \Delta')$
$\Gamma', \varphi \wedge \psi; \Delta$	$A_p(\Gamma', \varphi; \Delta) \vee A_p(\Gamma', \psi; \Delta)$
$\Gamma', \varphi \vee \psi; \Delta$	$A_p(\Gamma', \varphi; \Delta) \wedge A_p(\Gamma', \psi; \Delta)$
$\Gamma', \varphi \cdot \psi; \Delta$	$A_p(\Gamma', \varphi, \psi; \Delta)$
$\Gamma; \varphi + \psi, \Delta'$	$A_p(\Gamma; \varphi, \psi, \Delta')$
$\Gamma_1, \Gamma_2; \varphi \cdot \psi, \Delta_1, \Delta_2$	$A_p(\Gamma_1; \varphi, \Delta_1) \cdot A_p(\Gamma_2; \psi, \Delta_2)$
$\Gamma_1, \Gamma_2, \varphi + \psi; \Delta_1, \Delta_2$	$A_p(\Gamma_1, \varphi; \Delta_1) \cdot A_p(\Gamma_2, \psi; \Delta_2)$
$\Gamma_1, \Gamma_2; \Delta_1, \Delta_2$	$A_p(\Gamma_1; \Delta_1) + A_p(\Gamma_2; \Delta_2)$
$\Gamma; \neg\varphi, \Delta'$	$A_p(\varphi, \Gamma; \Delta')$
$\Gamma', \neg\varphi; \Delta$	$A_p(\Gamma'; \varphi, \Delta)$
otherwise	$\bot$

Γ_1, Γ_2 is equal to Γ'. On the other hand, in the latter case, both sequents $\Gamma_1, A_p(\Gamma_1; \varphi) \Rightarrow \varphi$ and $\Gamma_2, \psi \Rightarrow E_p(\Gamma_2, \psi)$ are provable by the hypothesis of induction, and hence the sequent $\Gamma_1, A_p(\Gamma_1; \varphi), \Gamma_2, \varphi\backslash\psi \Rightarrow E_p(\Gamma_2, \psi)$ is provable in **FL**. But, then we can get the provability of $\Gamma_1, \Gamma_2, \varphi\backslash\psi \Rightarrow A_p(\Gamma_1; \varphi)\backslash E_p(\Gamma_2, \psi)$ only when Γ_1 is empty and hence Γ' is equal to Γ_2. In addition to this, there remains an exceptional case *when* $A_p(\Gamma_1; \varphi)$ *is provable* in **FL**. In fact, in this case $\Gamma_1, \Gamma_2, \varphi\backslash\psi \Rightarrow E_p(\Gamma_2, \psi)$ is provable, and therefore $E_p(\Gamma_2, \psi)$ must be also put into $\mathcal{E}_p(\Gamma)$.

Table 5. *Modification of* $A_p(\Gamma;\,\Delta)$ *for* **InFL$_{\mathbf{ew}}$**

$\Gamma;\,\Delta$ matches	$\mathcal{A}_p(\Gamma;\,\Delta)$ contains
$\Gamma', r; r, \Delta'$	$\top$
$\Gamma', q; \Delta$	$\neg q$
$\Gamma; q, \Delta'$	q

Thus, we need a certain modifications of E_p and A_p for noncommutative substructural logics. In the following, we will define operations E_p and A_p *with side conditions* for **FL** and show the uniform interpolation property of **FL** by using them. At the end of the present section, we show that once we assume the exchange rule, side conditions can be removed from our tables, since our definition of E_p and A_p with side conditions is equivalent to that of E_p and A_p for **FL$_{\mathbf{e}}$**. Hence, E_p and A_p for **FL** can be regarded as a natural generalization of the definition of them for **FL$_{\mathbf{e}}$**.

We define now formulas $E_p(\Gamma)$ and $A_p(\Gamma \mid \Delta\ ;\alpha)$ in the following Tables from 6 to 8, where p is any propositional variable, α a formula or empty and Γ and Δ are (possibly empty) finite sequences of formulas, and show Theorem 5.2. Because of noncommutativity of **FL**, A_p must be defined for these triples $\langle\Gamma, \Delta, \alpha\rangle$ (cf. Lemma 3.1). Similarly as commutative cases, sets of formulas $\mathcal{E}_p(\Gamma)$ and $\mathcal{A}_p(\Gamma \mid \Delta; \alpha)$ are defined first, and then formulas $E_p(\Gamma)$ and $A_p(\Gamma \mid \Delta; \alpha)$ are defined to be the conjunction of all formulas in $\mathcal{E}_p(\Gamma)$ and the disjunction of all formulas in $\mathcal{A}_p(\Gamma \mid \Delta; \alpha)$, respectively. It can be verified easily that sets $\mathcal{E}_p(\Gamma)$ and $\mathcal{A}_p(\Gamma \mid \Delta; \alpha)$ are finite, and that both $E_p(\Gamma)$ and $A_p(\Gamma \mid \Delta; \alpha)$ are defined by simultaneous induction on the weight of sequences Γ and Γ, Δ, α, respectively. In these tables, we assume that p is a fixed propositional variable under consideration, q is either an arbitrary propositional variable other than p or any constant, and r is any propositional variable.

Different from commutative cases, sometimes the definition of $\mathcal{E}_p$ and $\mathcal{A}_p$ depends on whether their side conditions are provable or not in **FL**. In Table 6 for E_p, some rows are expressed as "$\mathcal{E}_p(\Gamma)$ contains the formula X with a side condition Y", which means that the set $\mathcal{E}_p(\Gamma)$ contains the formula X when Y is provable in **FL**. For instance, if Γ matches $\Sigma_1, \Sigma_2, \varphi\backslash\psi, \Delta$, then $\mathcal{E}_p(\Gamma)$ contains $E_p(\Sigma_1, \psi, \Delta)$ as long as $A_p(\emptyset \mid \Sigma_2\ ;\ \varphi)$ is provable in **FL**. The same notation is used also in Table 3 for A_p. More importantly, *the presence of side conditions does not cause any problem of well-definedness of* E_p *and* A_p *because of the decidability of* **FL**, shown in Corollary 2.2.

Lemma 5.1. *Both* $E_p(\Gamma)$ *and* $A_p(\Gamma \mid \Delta; \alpha)$ *are well-defined, and are obtained from* Γ, Δ, α *in a constructive way.*

Theorem 5.2. *Let* Γ, Π *and* Δ *be finite sequences of formulas and* α *be a single formula or empty. For every propositional variable* p *there exist formulas* $E_p(\Pi)$ *and* $A_p(\Gamma \mid \Delta; \alpha)$ *such that*

(1) (a) $V(E_p(\Pi)) \subseteq V(\Pi)\backslash\{p\}$,
    (b) $V(A_p(\Gamma \mid \Delta; \alpha)) \subseteq V(\Gamma, \Delta, \alpha)\backslash\{p\}$.

Table 6. *Definition of* $E_p(\Gamma)$ *for* **FL**

Γ matches	$\mathcal{E}_p(\Gamma)$ contains	side conditions
$\emptyset$	1	
p	$\top$	
q	q	
$\Sigma, \varphi \wedge \psi, \Delta$	$E_p(\Sigma, \varphi, \Delta) \wedge E_p(\Sigma, \psi, \Delta)$	
$\Sigma, \varphi \vee \psi, \Delta$	$E_p(\Sigma, \varphi, \Delta) \vee E_p(\Sigma, \psi, \Delta)$	
$\Sigma, \varphi \cdot \psi, \Delta$	$E_p(\Sigma, \varphi, \psi, \Delta)$	
$\Sigma, \varphi\backslash\psi, \Delta$	$A_p(\emptyset \mid \Sigma\ ;\ \varphi)\backslash E_p(\psi, \Delta)$	
$\Sigma_1, \Sigma_2, \varphi\backslash\psi, \Delta$	$E_p(\Sigma_1, \psi, \Delta)$	$[A_p(\emptyset \mid \Sigma_2\ ;\ \varphi)]$
$\Sigma, \psi/\varphi, \Delta$	$E_p(\Sigma, \psi)/A_p(\Delta \mid \emptyset;\ \varphi)$	
$\Sigma, \psi/\varphi, \Delta_1, \Delta_2$	$E_p(\Sigma, \psi, \Delta_2)$	$[A_p(\Delta_1 \mid \emptyset\ ;\ \varphi)]$
Γ_1, Γ_2	$E_p(\Gamma_1) \cdot E_p(\Gamma_2)$	$(\Gamma_1 \neq \emptyset, \Gamma_2 \neq \emptyset)$
otherwise	$\top$	

(2) (a) $\Pi \Rightarrow E_p(\Pi)$ *is provable in* **FL**,

(b) $\Gamma, A_p(\Gamma \mid \Delta; \alpha), \Delta \Rightarrow \alpha$ *is provable in* **FL**.

(3) *Suppose that* $\Gamma, \Pi, \Delta \Rightarrow \alpha$ *is provable in* **FL**. *Then*

(a) $\Gamma, E_p(\Pi), \Delta \Rightarrow \alpha$ *is provable in* **FL**, *when* $p \notin V(\Gamma, \Delta, \alpha)$,

(b) $\Pi \Rightarrow A_p(\Gamma \mid \Delta; \alpha)$ *is provable in* **FL**, *when* $p \notin V(\Pi)$

Proof. It is obvious that the statements 1(a) and 1(b) hold. We notice here that the remaining statements can be shown similarly as for the case of $\mathbf{FL_e}$, as long as Π, Γ and Δ are concerned only with propositional variables, constants and logical connectives $\vee$ and $\wedge$. For, their behaviors are not essentially affected by the presence or the absence of exchange rule (compare Tables from 6 to 8 for **FL** with Tables 1 and 2 for $\mathbf{FL_e}$).

Now, consider the second statements 2(a) and 2(b). They can be shown by using simultaneous induction on the weight. Let us consider 2(a) for the case where Π is equal to $\Sigma, \varphi\backslash\psi, \Delta$. (It will be interesting to compare the following proof with the corresponding part in the proof of Theorem 3.5. It will be seen that by the lack of exchange rule, E_p and A_p will take their default values $\top$ and $\bot$, respectively, for many cases, by which statements 2(a) and 2(b) become trivially true.) We divide Σ into Σ_1, Σ_2. By the hypothesis of induction, both $\Sigma_1, \psi, \Delta \Rightarrow E_p(\Sigma_1, \psi, \Delta)$ and $A_p(\emptyset \mid \Sigma_2; \varphi), \Sigma_2 \Rightarrow \varphi$ are provable. Thus, $\Sigma_1, A_p(\emptyset \mid \Sigma_2; \varphi), \Sigma_2, \varphi\backslash\psi, \Delta \Rightarrow E_p(\Sigma_1, \psi, \Delta)$ is also provable. Suppose first that Σ_1 is empty (and hence $\Sigma_2 = \Sigma$). Then, by applying the rule $(\Rightarrow, \backslash)$ we get

https://doi.org/10.1017/S175502031400015X

Table 7. *Definition of* $A_p(\Gamma \mid \Delta; \alpha)$ *for* **FL**

$\Gamma \mid \Delta; \alpha$ matches	$\mathcal{A}_p(\Gamma \mid \Delta; \alpha)$ contains
$\emptyset \mid \emptyset; \emptyset$	0
$0 \mid \emptyset; \emptyset$	1
$\emptyset \mid 0; \emptyset$	1
$\emptyset \mid \emptyset; q$	q
$r \mid \emptyset; r$	1
$\emptyset \mid r; r$	1
$\Gamma \mid \Delta; 0$	$A_p(\Gamma \mid \Delta; \emptyset)$
$\Gamma \mid \Delta; \top$	$\top$
$\Gamma_1, 1, \Gamma_2 \mid \Delta; \alpha$	$A_p(\Gamma_1, \Gamma_2 \mid \Delta; \alpha)$
$\Gamma \mid \Delta_1, 1, \Delta_2; \alpha$	$A_p(\Gamma \mid \Delta_1, \Delta_2; \alpha)$
$\Gamma_1, \bot, \Gamma_2 \mid \Delta; \alpha$	$\top$
$\Gamma \mid \Delta_1, \bot, \Delta_2; \alpha$	$\top$
$\Gamma_1, \varphi \wedge \psi, \Gamma_2 \mid \Delta; \alpha$	$A_p(\Gamma_1, \varphi, \Gamma_2 \mid \Delta; \alpha) \vee A_p(\Gamma_1, \psi, \Gamma_2 \mid \Delta; \alpha)$
$\Gamma \mid \Delta_1, \varphi \wedge \psi, \Delta_2; \alpha$	$A_p(\Gamma \mid \Delta_1, \varphi, \Delta_2; \alpha) \vee A_p(\Gamma \mid \Delta_1, \psi, \Delta_2; \alpha)$
$\Gamma_1, \varphi \vee \psi, \Gamma_2 \mid \Delta; \alpha$	$A_p(\Gamma_1, \varphi, \Gamma_2 \mid \Delta; \alpha) \wedge A_p(\Gamma_1, \psi, \Gamma_2 \mid \Delta; \alpha)$
$\Gamma \mid \Delta_1, \varphi \vee \psi, \Delta_2; \alpha$	$A_p(\Gamma \mid \Delta_1, \varphi, \Delta_2; \alpha) \wedge A_p(\Gamma \mid \Delta_1, \psi, \Delta_2; \alpha)$
$\Gamma \mid \Delta; \varphi \wedge \psi$	$A_p(\Gamma \mid \Delta; \varphi) \wedge A_p(\Gamma \mid \Delta; \psi)$
$\Gamma \mid \Delta; \varphi \vee \psi$	$A_p(\Gamma \mid \Delta; \varphi) \vee A_p(\Gamma \mid \Delta; \psi)$

that $\Sigma, \varphi\backslash\psi, \Delta \Rightarrow A_p(\emptyset \mid \Sigma_2; \varphi)\backslash E_p(\Sigma_1, \psi, \Delta)$. Note that the formula $A_p(\emptyset \mid \Sigma_2; \varphi)\backslash E_p(\Sigma_1, \psi, \Delta)$ belongs to $\mathcal{E}_p(\Sigma, \varphi\backslash\psi, \Delta)$. Suppose next that $A_p(\emptyset \mid \Sigma_2; \varphi)$ is provable. Then, applying cut rule, we have that $\Sigma_1, \Sigma_2, \varphi\backslash\psi, \Delta \Rightarrow E_p(\Sigma_1, \psi, \Delta)$ is provable, and $E_p(\Sigma_1, \psi, \Delta)$ belongs to $\mathcal{E}_p(\Sigma, \varphi\backslash\psi, \Delta)$. For other cases, $\top$ belongs to $\mathcal{E}_p(\Sigma, \varphi\backslash\psi, \Delta)$. Thus, $\Sigma, \varphi\backslash\psi, \Delta \Rightarrow E_p(\Sigma, \varphi\backslash\psi, \Delta)$ is provable.

Consider next 2(b) for the case where a formula $\varphi\backslash\psi$ appears in Γ or Δ in $A_p(\Gamma \mid \Delta; \alpha)$. Suppose first that Γ is $\Gamma_1, \Gamma_2, \varphi\backslash\psi, \Gamma_3$. By the hypothesis of induction, both $\Gamma_2, A_p(\Gamma_2 \mid \emptyset\ ; \varphi) \Rightarrow \varphi$ and $\Gamma_1, \psi, \Gamma_3, A_p(\Gamma_1, \psi, \Gamma_3 | \Delta; \alpha), \Delta \Rightarrow \alpha$ are provable. Then, $\Gamma_1, \Gamma_2, A_p(\Gamma_2 | \emptyset\ ; \varphi), \varphi\backslash\psi, \Gamma_3, A_p(\Gamma_1, \psi, \Gamma_3 | \Delta; \alpha), \Delta \Rightarrow \alpha$ is provable. So, when $A_p(\Gamma_2 \mid \emptyset\ ; \varphi)$ is provable, $\Gamma_1, \Gamma_2, \varphi\backslash\psi, \Gamma_3, A_p(\Gamma_1, \psi, \Gamma_3 \mid \Delta; \alpha), \Delta \Rightarrow \alpha$ is

https://doi.org/10.1017/S175502031400015X

Table 8. *Definition of* $A_p(\Gamma \mid \Delta; \alpha)$ *for* **FL**

$\Gamma \mid \Delta; \alpha$ matches	$\mathcal{A}_p(\Gamma \mid \Delta; \alpha)$ contains	side conditions
$\Gamma_1, \varphi \cdot \psi, \Gamma_2 \mid \Delta; \alpha$	$A_p(\Gamma_1, \varphi, \psi, \Gamma_2 \mid \Delta; \alpha)$	
$\Gamma \mid \Delta_1, \varphi \cdot \psi, \Delta_2; \alpha$	$A_p(\Gamma \mid \Delta_1, \varphi, \psi, \Delta_2; \alpha)$	
$\Gamma \mid \Delta; \varphi \cdot \psi$	$A_p(\Gamma \mid \emptyset; \varphi) \cdot A_p(\emptyset \mid \Delta; \psi)$	
$\Gamma_1, \Gamma_2 \mid \Delta; \varphi \cdot \psi$	$A_p(\Gamma_2 \mid \Delta; \psi)$	$[A_p(\Gamma_1 \mid \emptyset; \varphi)]$
$\Gamma \mid \Delta_1, \Delta_2; \varphi \cdot \psi$	$A_p(\Gamma \mid \Delta_1; \varphi)$	$[A_p(\emptyset \mid \Delta_2; \psi)]$
$\Gamma \mid \Delta_1, \varphi\backslash\psi, \Delta_2; \alpha$	$A_p(\Gamma \mid \psi, \Delta_2; \alpha) \cdot A_p(\emptyset \mid \Delta_1; \varphi)$	
$\Gamma_1, \Gamma_2 \mid \Delta_1, \varphi\backslash\psi, \Delta_2; \alpha$	$A_p(\Gamma_2 \mid \Delta_1; \varphi)$	$[A_p(\Gamma_1 \mid \psi, \Delta_2; \alpha)]$
$\Gamma \mid \Delta_0, \Delta_1, \varphi\backslash\psi, \Delta_2; \alpha$	$A_p(\Gamma \mid \Delta_0, \psi, \Delta_2; \alpha)$	$[A_p(\emptyset \mid \Delta_1; \varphi)]$
$\Gamma_1, \Gamma_2, \varphi\backslash\psi, \Gamma_3 \mid \Delta; \alpha$	$A_p(\Gamma_1, \psi, \Gamma_3 \mid \Delta; \alpha)$	$[A_p(\Gamma_2 \mid \emptyset; \varphi)]$
$\Gamma_1, \psi/\varphi, \Gamma_2 \mid \Delta; \alpha$	$A_p(\Gamma_2 \mid \emptyset; \varphi) \cdot A_p(\Gamma_1, \psi \mid \Delta; \alpha)$	
$\Gamma_1, \psi/\varphi, \Gamma_2 \mid \Delta_1, \Delta_2; \alpha$	$A_p(\Gamma_2 \mid \Delta_1; \varphi)$	$[A_p(\Gamma_1, \psi \mid \Delta_2; \alpha)]$
$\Gamma_1, \psi/\varphi, \Gamma_2, \Gamma_3 \mid \Delta; \alpha$	$A_p(\Gamma_1, \psi, \Gamma_3 \mid \Delta; \alpha)$	$[A_p(\Gamma_2 \mid \emptyset; \varphi)]$
$\Gamma \mid \Delta_1, \psi/\varphi, \Delta_2, \Delta_3; \alpha$	$A_p(\Gamma \mid \Delta_1, \psi, \Delta_3; \alpha)$	$[A_p(\emptyset \mid \Delta_2; \varphi)]$
$\Gamma \mid \Delta; \varphi\backslash\psi$	$A_p(\varphi, \Gamma \mid \Delta; \psi)$	
$\Gamma \mid \Delta; \psi/\varphi$	$A_p(\Gamma \mid \Delta, \varphi; \psi)$	
$\Gamma_1, \Gamma_2 \mid \Delta; \alpha$	$E_p(\Gamma_2)\backslash A_p(\Gamma_1 \mid \Delta; \alpha)$	$(\Gamma_2 \neq \emptyset)$
$\Gamma \mid \Delta_1, \Delta_2; \alpha$	$A_p(\Gamma \mid \Delta_2; \alpha)/E_p(\Delta_1)$	$(\Delta_1 \neq \emptyset)$
otherwise	$\bot$	

provable, where $A_p(\Gamma_1, \psi, \Gamma_3 | \Delta; \alpha) \in \mathcal{A}_p(\Gamma \mid \Delta; \alpha)$. Next suppose that Γ is Γ_1, Γ_2 and Δ is $\Delta_0, \Delta_1, \varphi\backslash\psi, \Delta_2$. By the hypothesis, both $\Gamma_2, A_p(\Gamma_2 \mid \Delta_1; \varphi), \Delta_1 \Rightarrow \varphi$ and $\Gamma_1, A_p(\Gamma_1 | \Delta_0, \psi, \Delta_2; \alpha), \Delta_0, \psi, \Delta_2 \Rightarrow \alpha$ are provable. Hence, $\Gamma_1, A_p(\Gamma_1 \mid \Delta_0, \psi, \Delta_2; \alpha), \Delta_0, \Gamma_2, A_p(\Gamma_2 \mid \Delta_1; \varphi), \Delta_1, \varphi\backslash\psi, \Delta_2 \Rightarrow \alpha$ is provable. We consider the following three cases. First we assume that $\Gamma_2 = \emptyset$, which implies $\Gamma_1 = \Gamma$. If moreover $\Delta_0 = \emptyset$ then we have that $\Gamma, A_p(\Gamma|\psi, \Delta_2; \alpha) \cdot A_p(\emptyset|\Delta_1; \varphi), \Delta_1, \varphi\backslash\psi, \Delta_2 \Rightarrow \alpha$ is provable. Recall that $A_p(\Gamma| \psi, \Delta_2; \alpha) \cdot A_p(\emptyset|\Delta_1; \varphi)$ belongs to $\mathcal{A}_p(\Gamma|\Delta_1, \varphi\backslash\psi, \Delta_2; \alpha)$. On the other hand, if $A_p(\emptyset \mid \Delta_1; \varphi)$ is provable then we get $\Gamma, A_p(\Gamma|\Delta_0, \psi, \Delta_2; \alpha), \Delta_0, \Delta_1, \varphi\backslash\psi, \Delta_2 \Rightarrow \alpha$, where $A_p(\Gamma \mid \Delta_0, \psi, \Delta_2; \alpha)$ belongs to $\mathcal{A}_p(\Gamma \mid \Delta_0, \Delta_1, \varphi\backslash\psi, \Delta_2; \alpha)$. Next we assume that $\Gamma_2 \neq \emptyset$. In this case, if $\Delta_0 = \emptyset$ and $A_p(\Gamma_1 \mid \psi, \Delta_2; \alpha)$ is provable, then we

https://doi.org/10.1017/S175502031400015X

have $\Gamma_1, \Gamma_2, A_p(\Gamma_2|\Delta_1;\varphi), \Delta_1, \varphi\backslash\psi, \Delta_2 \Rightarrow \alpha$, where $A_p(\Gamma_2|\Delta_1;\varphi)$ belongs to $\mathcal{A}_p(\Gamma_1, \Gamma_2|\Delta_1, \varphi\backslash\psi, \Delta_2;\alpha)$.

We show the third statements 3(a) and 3(b) by using the weight of the sequent $\Gamma, \Pi, \Delta \Rightarrow \alpha$. They can be shown easily when the sequent is an initial sequent of **FL**. Otherwise, the sequent is obtained by applying one of rules, let us say (R), of **FL**. We will consider here the case where (R) is $(\backslash \Rightarrow)$, for example, and give the detailed proof. We suppose that $\Gamma, \Pi, \Delta \Rightarrow \alpha$ is equal to $\Theta, \Sigma, \varphi\backslash\psi, \Lambda \Rightarrow \alpha$, and is obtained in the following way:

$$\frac{\Sigma \Rightarrow \varphi \quad \Theta, \psi, \Lambda \Rightarrow \alpha}{\Theta, \Sigma, \varphi\backslash\psi, \Lambda \Rightarrow \alpha}\ (\backslash \Rightarrow)$$

Since the weight of each of upper sequents is smaller than that of the lower sequent, we can assume that statements 3(a) and 3(b) hold for both of them. To show statements 3(a) and 3(b) for the sequent $\Theta, \Sigma, \varphi\backslash\psi, \Lambda \Rightarrow \alpha$, we need to consider several cases depending on which part of $\Theta, \Sigma, \varphi\backslash\psi, \Lambda$ can be regarded as Π of our Theorem. In the following, we assume always that the propositional variable p will never occur in any formula outside of Π when we consider 3(a), but p will never occur in any formula inside of Π when we consider 3(b).

(1) Case where Π is a part of Θ, Σ. There are the following three subcases.
(1-1) Π is a part of Θ. That is, Θ is of the form Θ_1, Π, Θ_2, where both Θ_1 and Θ_2 may be empty. For 3(a), since $\Theta_1, \Pi, \Theta_2, \psi, \Lambda \Rightarrow \alpha$ is provable, $\Theta_1, E_p(\Pi), \Theta_2, \psi, \Lambda \Rightarrow \alpha$ is also provable by the hypothesis of induction. Moreover, as $\Sigma \Rightarrow \varphi$ is provable, we have that $\Theta_1, E_p(\Pi), \Theta_2, \Sigma, \varphi\backslash\psi, \Lambda \Rightarrow \alpha$ is provable. Thus, 3(a) is shown. For 3(b), since $\Theta_1, \Pi, \Theta_2, \psi, \Lambda \Rightarrow \alpha$ is provable, $\Pi \Rightarrow A_p(\Theta_1 \mid \Theta_2, \psi, \Lambda; \alpha)$ is provable by the hypothesis of induction. On the other hand, as $\Sigma \Rightarrow \varphi$ is provable, $A_p(\emptyset \mid \Sigma; \varphi)$ is also provable. In such a case, $A_p(\Theta_1 \mid \Theta_2, \psi, \Lambda; \alpha)$ belongs to $\mathcal{A}_p(\Theta_1 \mid \Theta_2, \Sigma, \varphi\backslash\psi, \Lambda; \alpha)$. Thus, 3(b) is shown.
(1-2) Π is Π_1, Π_2 with nonempty Π_1 and Π_2 such that Θ is of the form Θ_0, Π_1 and Σ is of the form Π_2, Σ_0: For 3(a), since both $\Pi_2, \Sigma_0 \Rightarrow \varphi$ and $\Theta_0, \Pi_1, \psi, \Lambda \Rightarrow \alpha$ are provable, $E_p(\Pi_2), \Sigma_0 \Rightarrow \varphi$ and $\Theta_0, E_p(\Pi_1), \psi, \Lambda \Rightarrow \alpha$ are also provable by the hypothesis of induction. Thus, $\Theta_0, E_p(\Pi_1) \cdot E_p(\Pi_2), \Sigma_0, \varphi\backslash\psi, \Lambda \Rightarrow \alpha$, and hence $\Theta_0, E_p(\Pi), \Sigma_0, \varphi\backslash\psi, \Lambda \Rightarrow \alpha$ are provable. For 3(b), by the hypothesis of induction, both $\Pi_1 \Rightarrow A_p(\Theta_0 \mid \psi, \Lambda; \alpha)$ and $\Pi_2 \Rightarrow A_p(\emptyset \mid \Sigma_0; \varphi)$ are provable. Therefore $\Pi \Rightarrow A_p(\Theta_0 \mid \psi, \Lambda; \alpha) \cdot A_p(\emptyset \mid \Sigma_0; \varphi)$ is provable, which implies 3(b).
(1-3) Π is a part of Σ. Similarly as (1-1).

(2) Case where Π contains the principal formula $\varphi\backslash\psi$ of this $(\backslash \Rightarrow)$. Consider the following two subcases.
(2-1) Π includes also Σ. Then, we can assume that Π is $\Pi_1, \Pi_2, \varphi\backslash\psi, \Pi_3$ such that (i) Θ is of the form Θ_0, Π_1, (ii) $\Sigma = \Pi_2$ and (iii) Λ is of the form Π_3, Λ_0. Here, Π_1 may be empty, and either Π_3 or Λ_0 may be empty. For 3(a), by the hypothesis of induction, $\Theta_0, E_p(\Pi_1, \psi, \Pi_3), \Lambda_0 \Rightarrow \alpha$ is provable. On the other hand, as $\Pi_2 \Rightarrow \varphi$ is provable, $A_p(\emptyset \mid \Pi_2; \varphi)$ is provable. Thus 3(a) holds, since $E_p(\Pi_1, \psi, \Pi_3)$ belongs to $\mathcal{E}_p(\Pi_1, \Pi_2, \varphi\backslash\psi, \Pi_3)$ in this case. The statement 3(b) is almost immediate.
(2-2) Otherwise. Then Π is $\Pi_1, \varphi\backslash\psi, \Pi_2$ such that Σ is of the form Σ_0, Π_1 with nonempty Σ_0 and Λ is of the form Π_2, Λ_0. For 3(a), by the hypothesis of induction $\Sigma_0 \Rightarrow A_p(\emptyset \mid \Pi_1; \varphi)$ and $\Theta, E_p(\psi, \Pi_2), \Lambda_0 \Rightarrow \alpha$ are provable. Hence, $\Theta, \Sigma_0, A_p(\emptyset \mid \Pi_1; \varphi)\backslash E_p(\psi, \Pi_2), \Lambda_0 \Rightarrow \alpha$ is provable, where $A_p(\emptyset \mid \Pi_1; \varphi)\backslash E_p(\psi, \Pi_2)$ belongs to $\mathcal{E}_p(\Pi_1, \varphi\backslash\psi, \Pi_2)$. Thus, we have 3(a). For 3(b), by the hypothesis of induction, both $E_p(\Sigma_0)$,

https://doi.org/10.1017/S175502031400015X

$\Pi_1 \Rightarrow \varphi$ and $\psi, \Pi_2 \Rightarrow A_p(\Theta \mid \Lambda_0; \alpha)$ are provable. Thus, $E_p(\Sigma_0), \Pi_1, \varphi\backslash\psi, \Pi_2 \Rightarrow A_p(\Theta \mid \Lambda_0; \alpha)$ is provable, and hence $\Pi_1, \varphi\backslash\psi, \Pi_2 \Rightarrow E_p(\Sigma_0)\backslash A_p(\Theta \mid \Lambda_0; \alpha)$ is provable. From this 3(b) follows.

(3) Case where Π is a part of Λ. The proof is similar to the case (1-1). □

COROLLARY 5.3. *The uniform interpolation property holds for full Lambek propositional calculus* **FL***.*

Before concluding this section, we will explain how Tables 1 and 2 of E_p and A_p for $\mathbf{FL_e}$ can be obtained naturally from Tables from 6 to 8 if we assume exchange rule, in addition. Let us consider our Tables for **FL**. First, we replace both formulas $\varphi\backslash\psi$ and ψ/φ by $\varphi \to \psi$, and moreover regard every finite sequence Γ of formulas as a multiset of formulas. In this way, the formula $E_p(\Gamma, \varphi\backslash\psi, \Delta)$ in our Table 6, for example, will be translated into a formula $E_p(\Gamma, \Delta, \varphi \to \psi)$ in $\mathbf{FL_e}$. We translate also the set $\mathcal{A}_p(\Gamma \mid \Delta; \alpha)$ and hence the formula $A_p(\Gamma \mid \Delta; \alpha)$ in Tables 7 and 8 into a new definition of $\mathcal{A}_p(\Gamma, \Delta; \alpha)$ and $A_p(\Gamma, \Delta; \alpha)$, respectively. We compare this new tables for $\mathbf{FL_e}$ with the tables for $\mathbf{FL_e}$ in Section 3. It is easily seen that these formulas E_p and A_p of $\mathbf{FL_e}$ obtained in this way coincide exactly with those in the tables for $\mathbf{FL_e}$ in Section 3, as long as they are defined without using any side condition.

We consider now the following two rows in Table 6 for $\mathcal{E}_p$, where the second one has a side condition.

1) $\Sigma, \varphi\backslash\psi, \Delta$ $\quad A_p(\emptyset \mid \Sigma\ ;\ \varphi)\backslash E_p(\psi, \Delta)$,
2) $\Sigma_1, \Sigma_2, \varphi\backslash\psi, \Delta$ $\quad E_p(\Sigma_1, \psi, \Delta)$ $\quad$ with the side condition $A_p(\emptyset \mid \Sigma_2\ ;\ \varphi)$.

By the above translation we get the following new tables concerned with the implication;

1') $\Sigma, \Delta, \varphi \to \psi$ $\quad A_p(\Sigma; \varphi) \to E_p(\Delta, \psi)$,
2') $\Sigma_1, \Sigma_2, \Delta, \varphi \to \psi$ $\quad E_p(\Sigma_1, \Delta, \psi)$ $\quad$ with a condition $A_p(\Sigma_2; \varphi)$.

Take a multiset of formulas of the form $\Theta, \Lambda, \varphi \to \psi$. Then $\mathcal{E}_p(\Theta, \Lambda, \varphi \to \psi)$ contains $A_p(\Theta; \varphi) \to E_p(\Lambda, \psi)$ by 1'), and also $E_p(\Lambda, \psi)$ when $A_p(\Theta; \varphi)$ is provable in $\mathbf{FL_e}$. Recall that the formula $E_p(\Theta, \Lambda, \varphi \to \psi)$ is obtained by taking the conjunction of all formulas in $\mathcal{E}_p(\Theta, \Lambda, \varphi \to \psi)$, and hence it is of the form $\chi \wedge (A_p(\Theta; \varphi) \to E_p(\Lambda, \psi)) \wedge E_p(\Lambda, \psi)$, when $A_p(\Theta; \varphi)$ is provable. But, obviously, $A_p(\Theta; \varphi) \to E_p(\Lambda, \psi)$ implies $E_p(\Lambda, \psi)$, when $A_p(\Theta; \varphi)$ is provable, and hence $E_p(\Theta, \Lambda, \varphi \to \psi)$ is equivalent to a formula $\chi \wedge (A_p(\Theta; \varphi) \to E_p(\Lambda, \psi))$. Thus, 2') becomes redundant in the definition of $\mathcal{E}_p$ in the case of $\mathbf{FL_e}$, and we get the same formula E_p as defined for $\mathbf{FL_e}$.

Next we consider the following four rows in Table 8 for $\mathcal{A}_p$, in each of which a formula $\varphi \to \psi$ appears at the left side of semicolon.

1) $\Gamma \mid \Delta_1, \varphi\backslash\psi, \Delta_2; \alpha$ $\quad A_p(\Gamma \mid \psi, \Delta_2; \alpha) \cdot A_p(\emptyset \mid \Delta_1; \varphi)$,
2) $\Gamma_1, \Gamma_2 \mid \Delta_1, \varphi\backslash\psi, \Delta_2; \alpha$ $\quad A_p(\Gamma_2 \mid \Delta_1; \varphi)$ $\quad$ with $A_p(\Gamma_1 \mid \psi, \Delta_2; \alpha)$,
3) $\Gamma \mid \Delta_0, \Delta_1, \varphi\backslash\psi, \Delta_2; \alpha$ $\quad A_p(\Gamma \mid \Delta_0, \psi, \Delta_2; \alpha)$ $\quad$ with $A_p(\emptyset \mid \Delta_1; \varphi)$,
4) $\Gamma_1, \Gamma_2, \varphi\backslash\psi, \Gamma_3 \mid \Delta; \alpha$ $\quad A_p(\Gamma_1, \psi, \Gamma_3 \mid \Delta; \alpha)$ $\quad$ with $A_p(\Gamma_2 \mid \emptyset\ ;\ \varphi)$.

We translate them to get new tables of $\mathcal{A}_p$ for $\mathbf{FL_e}$, which can be summarized as follows. Note that the above 3) and 4) become essentially the same after the translation.

5) $\Gamma, \Delta, \varphi \to \psi; \alpha$ $\quad A_p(\Delta, \psi; \alpha) \cdot A_p(\Gamma; \varphi)$,
6) $\Gamma, \Delta, \varphi \to \psi; \alpha$ $\quad A_p(\Gamma; \varphi)$ $\quad$ with a condition $A_p(\Delta, \psi; \alpha)$,
7) $\Gamma, \Delta, \varphi \to \psi; \alpha$ $\quad A_p(\Delta, \psi; \alpha)$ $\quad$ with a condition $A_p(\Gamma; \varphi)$.

https://doi.org/10.1017/S175502031400015X

Since $A_p(\Gamma, \Delta, \varphi \to \psi; \alpha)$ is obtained by taking the disjunction of all formulas in $\mathcal{A}_p(\Gamma, \Delta, \varphi \to \psi; \alpha)$, it will take a form $\theta \vee (A_p(\Delta, \psi; \alpha) \cdot A_p(\Gamma; \varphi)) \vee A_p(\Gamma; \varphi) \vee A_p(\Delta, \psi; \alpha)$, in general. On the other hand, $A_p(\Gamma; \varphi)$ implies $A_p(\Delta, \psi; \alpha) \cdot A_p(\Gamma; \varphi)$ when $A_p(\Delta, \psi; \alpha)$ is provable, and also $A_p(\Delta, \psi; \alpha)$ implies $A_p(\Delta, \psi; \alpha) \cdot A_p(\Gamma; \varphi)$ when $A_p(\Gamma; \varphi)$ is provable. Hence, $A_p(\Gamma, \Delta, \varphi \to \psi; \alpha)$ is always equivalent to a formula $\theta \vee (A_p(\Delta, \psi; \alpha) \cdot A_p(\Gamma; \varphi))$. Other cases can be treated in the same way. Thus, our new tables of A_p for $\mathbf{FL_e}$ are essentially the same as that in Section 3.

§6. Propositional quantifiers and uniform interpolation. In this short section, we mention briefly substructural propositional logics with *propositional quantifiers* (or, second order substructural propositional logics) and uniform interpolation in substructural propositional logics. For each substructural propositional logic $\mathbf{L}$, define $\mathbf{L^2}$ be the substructural propositional logics with propositional quantifiers, i. e. with quantifications over propositional variables. (Note that different from the definition of $\mathbf{QL}$ in Montagna, 2012, we do not assume the following as an axiom of $\mathbf{L^2}$: $\forall p(\varphi \vee \xi) \to (\forall p\varphi \vee \xi)$, where p has no free occurrence in ξ.) By the use of existential and universal propositional quantifiers, uniform interpolation property of $\mathbf{L^2}$ can be shown immediately. But, in order to derive uniform interpolation property of $\mathbf{L}$ from this, we need to show that (1) $\mathbf{L^2}$ is *conservative* over $\mathbf{L}$ and (2) $\mathbf{L^2}$ has *quantifier elimination*, i.e., for each formula φ of $\mathbf{L}$ and each propositional variable p there exist formulas ϵ and α of $\mathbf{L}$ such that $\exists p\varphi$ ($\forall p\varphi$) is mutually equivalent to ϵ (α, respectively) in $\mathbf{L}$ (see, e.g., Montagna, 2012, where uniform interpolation in Δ-core fuzzy logics is discussed).

Here, for each substructural propositional logic $\mathbf{L}$ discussed so far, an interpretation of $\mathbf{L^2}$ into $\mathbf{L}$ can be given in just the same way as in Pitts (1992). The following theorem, which is a modification of Proposition 9 in Pitts (1992), says that the conservatively and quantifier elimination hold in $\mathbf{L^2}$ in our case. Now, let $\mathbf{L}$ be any of $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{InFL_e}$ and $\mathbf{InFL_{ew}}$.

To confirm that our proof can be carried out in the same way as that Proposition 9 in Pitts (1992), we need to check several things. First, it is easy to see that existentially quantified propositions $\exists p\varphi$ can be defined by $\forall q(\forall p(\varphi \to q) \to q)$ (with q not free in φ) in $\mathbf{FL_e}^2$. (But in $\mathbf{FL}$, it is necessary to replace this by $\forall q(\forall p(q/\varphi)\backslash q)$ (which is shown to be equivalent to $\forall q(q/\forall p(\varphi\backslash q))$).) The translation * of formulas of $\mathbf{L^2}$ to propositional formulas of $\mathbf{L}$ can be defined in the same way as that in Pitts (1992), i.e.,

(1) $r^* = r$ for every propositional variable or constant,
(2) $(\varphi \sharp \psi)^* = (\varphi^* \sharp \psi^*)$ for each $\sharp \in \{\wedge, \vee, \cdot, \backslash, /\}$,
(3) $(\forall p\, \varphi)^* = A_p(\varphi^*)$.

Here, $A_p(\varphi)$ means $A_p(\emptyset; \varphi)$ (or, $A_p(\emptyset|\emptyset; \varphi)$ for $\mathbf{FL}$). Obviously, φ^* is equal to φ for every propositional formula φ of $\mathbf{L}$. Then, results which correspond to Lemma 8 and Proposition 9 of Pitts (1992) can be shown for $\mathbf{L}$ and $\mathbf{L}^2$. We note that in proving Lemma 8 of Pitts (1992), the following *congruence property* of $\leftrightarrow$ in intuitionistic propositional logic $\mathbf{Int}$ was used;

$$\mathbf{Int} \vdash (\psi \leftrightarrow \psi') \Rightarrow \theta[\psi/q] \leftrightarrow \theta[\psi'/q]$$

But, this does not hold in general for logics under consideration. Instead, we can show the following modified congruence property in $\mathbf{FL}$ by replacing the provability with the *deducibility*;

$$\psi \leftrightarrow \psi' \vdash_{\mathbf{FL}} \theta[\psi/q] \leftrightarrow \theta[\psi'/q].$$

(For more information on the deducibility, see Galatos *et al.*, 2007.) As a matter of fact, the congruence property of this form is enough to get a result for **L** which corresponds to Lemma 8 of Pitts (1992). Thus, we have the following.

THEOREM 6.1. *Let* **L** *be any of* **FL**, $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{InFL_e}$ *and* $\mathbf{InFL_{ew}}$. *The translation* * *gives an interpretation of* $\mathbf{L}^2$ *into* **L** *such that (1) for every sequent* $\Gamma \Rightarrow \varphi$ *of a propositional logic* $\mathbf{L}^2$ *with propositional quantifiers, if* $\Gamma \Rightarrow \varphi$ *is provable in* $\mathbf{L}^2$ *then* $\Gamma^* \Rightarrow \varphi^*$ *is provable in* **L**, *and (2)* φ^* *is equal to* φ *for every propositional formula* φ *of* **L**.

§7. Uniform interpolation for substructural predicate logics. As mentioned in Henkin (1963), uniform interpolation holds neither for classical predicate logic nor for intuitionistic predicate logic. In the following, we show that uniform interpolation holds for each substructural predicate logic which is the minimum predicate extension of one of substructural propositional logics discussed in the present paper. We assume that *the language for predicate logics contains no function symbols*. First, we consider the case where the language contains neither individual constants, and then show how to modify the proof when it has individual constants. In the following, letters x, y, z, v etc. denote individual variables.

The sequent calculus **QFL** of full Lambek predicate calculus is obtained from **FL** by adding the following rules for quantifiers.

$$\frac{\Gamma, \alpha[v/x], \Delta \Rightarrow \varphi}{\Gamma, \forall x\alpha, \Delta \Rightarrow \varphi}\ (\forall \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha[y/x]}{\Gamma \Rightarrow \forall x\alpha}\ (\Rightarrow \forall)$$

$$\frac{\Gamma, \alpha[y/x], \Delta \Rightarrow \varphi}{\Gamma, \exists x\alpha, \Delta \Rightarrow \varphi}\ (\exists \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha[v/x]}{\Gamma \Rightarrow \exists x\alpha}\ (\Rightarrow \exists)$$

Here, $\alpha[v/x]$ and $\alpha[y/x]$ denote the formula obtained from α by replacing all free occurrences of x by v and y, respectively. Also, in the application of $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$, the individual variable y must satisfy the eigenvariable condition, i.e., it does not appear in the lower sequent as a free variable. Similarly, we can introduce minimum predicate extensions $\mathbf{QFL_e}$, $\mathbf{QFL_{ew}}$, $\mathbf{QInFL_e}$ and $\mathbf{QInFL_{ew}}$ of $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{InFL_e}$ and $\mathbf{InFL_{ew}}$, respectively. (Of course, for $\mathbf{QInFL_e}$ and $\mathbf{QInFL_{ew}}$, we need to replace φ by a multiset Φ in $(\forall \Rightarrow)$ and $(\exists \Rightarrow)$, and add a multiset Ψ to the right side of each sequent in $(\Rightarrow \forall)$ and $(\Rightarrow \exists)$.) The admissibility of cut rule can be extended to each minimum predicate extension (see, e.g., Dardžaniá, 1977; Grišin, 1982; Komori, 1986; Girard, 1987; Ono, 1990; Troelstra, 1992).

LEMMA 7.1. *Cut rule is admissible in each of* **QFL**, $\mathbf{QFL_{ew}}$, $\mathbf{QInFL_e}$ *and* $\mathbf{QInFL_{ew}}$.

Moreover, we can show the following (see Wang, 1963; Grišin, 1982; Komori, 1986). (In fact, this holds even if the language contains both individual constants and function symbols. See Kiriyama & Ono, 1991, for the details.)

THEOREM 7.2. *Each of structural predicate logics* **QFL**, $\mathbf{QFL_{ew}}$, $\mathbf{QInFL_e}$ *and* $\mathbf{QInFL_{ew}}$ *is decidable.*

The notion of the weight can be extended to first-order formulas, by defining that $w(\alpha) = 1$ for every atomic formula α and $w(\forall x\varphi) = w(\exists x\varphi) = w(\varphi) + 1$. We note that it is possible to formalize classical predicate logic in a cut-free sequent calculus without

https://doi.org/10.1017/S175502031400015X

"explicit" contraction rules, e.g., the one given by Kanger in Kanger (1957), the rule ($\forall \Rightarrow$) will be of the following form;

$$\frac{\Gamma, \forall x\alpha, \alpha[v/x] \Rightarrow \varphi}{\Gamma, \forall x\alpha \Rightarrow \varphi}$$

But the weight of the upper sequent becomes greater than the lower one. In the following, we discuss uniform interpolation property of $\mathbf{QFL_e}$ in details. For each *predicate symbol* p, we introduce the definition of $E_p(\Gamma)$ and $A_p(\Gamma; \alpha)$ in the following Table 9, in addition to Tables 1 and 2 for $\mathbf{FL_e}$. (To be precise, the 2nd and 3rd rows in Table 1 and the 3rd and 4th rows in Table 2 must be replaced by the 1st and 2nd rows, and the 5th and 6th rows in the following Table 9, respectively.) In Table 9, we assume the following:

- z is the first individual variable (in a fixed list of variables) which does not appear in Γ in the table for $\mathcal{E}_p(\Gamma)$ (and in $\Gamma; \alpha$ in the table for $\mathcal{A}_p(\Gamma; \alpha)$, respectively),
- v runs over all free variables appearing in Γ in the table for $\mathcal{E}_p(\Gamma)$ (and in $\Gamma; \alpha$ in the table for $\mathcal{A}_p(\Gamma; \alpha)$, respectively).

We can observe that every free variable in each member of $\mathcal{E}_p(\Gamma)$ ($\mathcal{A}_p(\Delta; \alpha)$) appears free in Γ (and Δ, α, respectively). There are only finitely many possible matchings in Tables 1, 2, and 9, which implies that both $\mathcal{E}_p(\Gamma)$ and $\mathcal{A}_p(\Delta; \alpha)$ are finite sets of formulas. Thus formulas $E_p(\Gamma)$ and $A_p(\Delta; \alpha)$ are well-defined. Now we show the following theorem which is an extension of Theorem 3.5. In the following, $V(\Delta)$ denotes the set of all *predicate symbols* appearing in a given multiset of formulas Δ.

Table 9. *Additional* $E_p(\Gamma)$ *and* $A_p(\Gamma; \alpha)$ *for* $\mathbf{QFL_e}$

Γ matches	$\mathcal{E}_p(\Gamma)$ contains
$p(v_1, \cdots, v_n)$	$\top$
$q(v_1, \cdots, v_k)$	$q(v_1, \cdots, v_k)$
$\Gamma', \forall x\beta$	$\bigwedge_v E_p(\Gamma', \beta[v/x]) \wedge \forall z E_p(\Gamma', \beta[z/x])$
$\Gamma', \exists x\beta$	$\exists z E_p(\Gamma', \beta[z/x])$
$\Gamma; \alpha$ matches	**$\mathcal{A}_p(\Gamma; \alpha)$ contains**
$\emptyset; q(v_1, \cdots, v_k)$	$q(v_1, \cdots, v_k)$
$r(v_1, \cdots, v_k); r(v_1, \cdots, v_k)$	1
$\Gamma; \forall x\beta$	$\forall z A_p(\Gamma; \beta[z/x])$
$\Gamma; \exists x\beta$	$\bigvee_v A_p(\Gamma; \beta[v/x]) \vee \exists z A_p(\Gamma; \beta[z/x])$
$\Gamma', \forall x\beta; \delta$	$\bigvee_v A_p(\Gamma', \beta[v/x]; \delta) \vee \exists z A_p(\Gamma', \beta[z/x]; \delta)$
$\Gamma', \exists x\beta; \delta$	$\forall z A_p(\Gamma', \beta[z/x]; \delta)$

THEOREM 7.3. *Let* Γ *be a multiset of first-order formulas, and* α *be a single first-order formula or empty. For every predicate symbol* p *there exist first-order formulas* $E_p(\Gamma)$ *and* $A_p(\Gamma;\alpha)$ *such that*

(1) (a) $V(E_p(\Gamma)) \subseteq V(\Gamma)\backslash\{p\}$,
 (b) $V(A_p(\Gamma;\alpha)) \subseteq V(\Gamma,\alpha)\backslash\{p\}$.
(2) (a) $\Gamma \Rightarrow E_p(\Gamma)$ *is provable in* $\mathbf{QFL_e}$,
 (b) $\Gamma, A_p(\Gamma;\alpha) \Rightarrow \alpha$ *is provable in* $\mathbf{QFL_e}$.
(3) *Let* Π *be any multiset of first-order formulas not containing the predicate symbol* p. *If* $\Pi, \Gamma \Rightarrow \alpha$ *is provable in* $\mathbf{QFL_e}$, *then*
 (a) $\Pi, E_p(\Gamma) \Rightarrow \alpha$ *is provable in* $\mathbf{QFL_e}$, *when* $p \notin V(\alpha)$,
 (b) $\Pi \Rightarrow A_p(\Gamma;\alpha)$ *is provable in* $\mathbf{QFL_e}$.

Proof. The proof goes essentially the same way as the proof of Theorem 3.5. For statements 2(a) and 2(b), we give here a proof for $\exists$. For 2(a), let us consider the case where Γ is of the form $\Delta, \exists x\beta$. Take the variable z, which does not appear in $\Delta, \exists x\beta$. By the hypothesis of induction, $\Delta, \beta[z/x] \Rightarrow E_p(\Delta, \beta[z/x])$ is provable in $\mathbf{QFL_e}$. Hence, $\Delta, \beta[z/x] \Rightarrow \exists z E_p(\Delta, \beta[z/x])$. Since z satisfies the eigenvariable condition, we have $\Delta, \exists x\beta \Rightarrow \exists z E_p(\Delta, \beta[z/x])$. From this, 2(a) in the present case follows.

For 2(b), consider first the case where $\Gamma;\alpha$ matches $\Gamma;\exists x\beta$. By the hypothesis of induction, we assume that $\Gamma, A_p(\Gamma;\beta[u/x]) \Rightarrow \beta[u/x]$ is provable, and hence $\Gamma, A_p(\Gamma;\beta[u/x]) \Rightarrow \exists x\beta$ is provable in $\mathbf{QFL_e}$ for every variable u. In particular, this holds for every variable which appears free in $\Gamma;\exists x\beta$ and also for the variable z. From the latter, the provability of $\Gamma, \exists z A_p(\Gamma;\beta[z/x]) \Rightarrow \exists x\beta$ follows. Thus, $\Gamma, \bigvee_v A_p(\Gamma;\beta[v/x]) \vee \exists z A_p(\Gamma;\beta[z/x]) \Rightarrow \exists x\beta$ is provable.

Next consider the case where $\Gamma;\alpha$ matches $\Delta, \exists x\beta;\delta$. Similarly to the above 2(a), we take the variable z. Then, $\Delta, \beta[z/x], A_p(\Delta, \beta[z/x];\delta) \Rightarrow \delta$ is provable in $\mathbf{QFL_e}$ by the hypothesis of induction. Hence $\Delta, \beta[z/x], \forall z A_p(\Delta, \beta[z/x];\delta) \Rightarrow \delta$ is provable. From this, the provability of $\Delta, \exists x\beta, \forall z A_p(\Delta, \beta[z/x];\delta) \Rightarrow \delta$ follows, as z does not appear in $\Delta, \exists x\beta;\delta$.

We discuss next 3(a) and 3(b). Suppose that the sequent $\Pi, \Gamma \Rightarrow \delta$ is provable, where Π is a multiset of formulas not containing the predicate symbol p. We give here a proof of the cases where the last rule applied is either ($\exists \Rightarrow$) or ($\Rightarrow \exists$). For ($\exists \Rightarrow$), it is necessary to consider the following two cases.

Case 1. Suppose first that Π is equal to $\Pi', \exists x\beta$. We show that the statement 3(a) holds, assuming that $p \notin V(\delta)$. Suppose that $\Pi, \Gamma \Rightarrow \delta$ follows from the upper sequent $\Pi', \beta[u/x], \Gamma \Rightarrow \delta$, where u is an individual variable which does not appear in $\Pi, \Gamma \Rightarrow \delta$ as a free variable. From the hypothesis of induction for 3(a), $\Pi', \beta[u/x], E_p(\Gamma) \Rightarrow \delta$ is provable in $\mathbf{QFL_e}$. Since u does not appear also in $E_p(\Gamma)$ as a free variable, we can show that $\Pi', \exists x\beta, E_p(\Gamma) \Rightarrow \delta$ is also provable. For 3(b), since $\Pi', \beta[u/x] \Rightarrow A_p(\Gamma;\delta)$ is provable by the hypothesis of induction, $\Pi', \exists x\beta \Rightarrow A_p(\Gamma;\delta)$ is provable as u does not appear in it as a free variable.

Case 2. Next we assume that Γ is equal to $\Gamma', \exists x\beta$ and show that 3(a), assuming that $p \notin V(\delta)$. Consider the case where the upper sequent is of the form $\Pi, \Gamma', \beta[u/x] \Rightarrow \delta$, where u does not appear in $\Pi, \Gamma \Rightarrow \delta$ as a free variable. By the hypothesis of induction, $\Pi, E_p(\Gamma', \beta[u/x]) \Rightarrow \delta$ is provable, and hence $\Pi, \exists u E_p(\Gamma', \beta[u/x]) \Rightarrow \delta$ is provable also. As neither of u and z appear in Γ as free variables, $\exists z E_p(\Gamma', \beta[z/x])$

https://doi.org/10.1017/S175502031400015X

$\Rightarrow \exists u E_p(\Gamma', \beta[u/x])$ is provable. Hence, $\Pi, \exists z E_p(\Gamma', \beta[z/x]) \Rightarrow \delta$ is provable. For 3(b), $\Pi \Rightarrow A_p(\Gamma', \beta[u/x]; \delta)$ is provable by the hypothesis of induction. Therefore, $\Pi \Rightarrow \forall u A_p(\Gamma', \beta[u/x]; \delta)$ is provable, and using the similar argument as above, $\Pi \Rightarrow \forall z A_p(\Gamma', \beta[z/x]; \delta)$ is shown to be provable. (Though z may appear in Π as a free variable, it does not matter.)

For $(\Rightarrow \exists)$, we may assume that the sequent $\Pi, \Gamma \Rightarrow \exists x\beta$ is derived from $\Pi, \Gamma \Rightarrow \beta[u/x]$ for some variable u by applying $(\Rightarrow \exists)$. When $p \notin V(\exists x\beta) = V(\beta[u/x])$, $\Pi, E_p(\Gamma) \Rightarrow \beta[u/x]$ is provable by the hypothesis, and thus $\Pi, E_p(\Gamma) \Rightarrow \exists x\beta$ is provable. Thus, 3(a) is shown. For 3(b), $\Pi \Rightarrow A_p(\Gamma; \beta[u/x])$ is provable by the hypothesis of induction. If u appears in $\Gamma, \exists x\beta$ then $\Pi \Rightarrow \bigvee_v A_p(\Gamma; \beta[v/x])$ is provable, where v runs over free variables appearing in $\Gamma, \exists x\beta$. Otherwise, $\Pi \Rightarrow \exists u A_p(\Gamma; \beta[u/x])$ is provable, and hence $\Pi \Rightarrow \exists z A_p(\Gamma; \beta[z/x])$ is provable as z does not appear in $\Gamma, \exists x\beta$. Then, the provability of $\Pi \Rightarrow \bigvee_v A_p(\Gamma; \beta[v/x]) \vee \exists z A_p(\Gamma; \beta[z/x])$ follows in either case. □

COROLLARY 7.4. *The uniform interpolation property holds for* $\mathbf{QFL_e}$ *without function symbols and individual constants.*

We remark that the 3rd row of $\mathcal{E}_p$, the 4th and the 5th rows of $\mathcal{A}_p$ in Table 9 are of the form different from the 4th row of $\mathcal{E}_p$, the 3rd and the 6th rows of $\mathcal{A}_p$. As is seen in the proof of Theorem 7.3, the latter corresponds to rules for quantifiers with eigenvariable condition while the former is not. One may suppose that each of the former cases could be simplified. For example, when $\Gamma; \alpha$ matches $\Gamma; \exists x\beta$, the formula $\bigvee_v A_p(\Gamma; \beta[v/x]) \vee \exists z A_p(\Gamma; \beta[z/x])$ in $\mathcal{A}_p$ would be simplified to the formula $\exists z A_p(\Gamma; \beta[z/x])$, since $\exists z A_p(\Gamma; \beta[z/x])$ will follows from $A_p(\Gamma; \beta[v/x])$ for any v. But this is not the case. For instance, consider the case where Γ is $p(v)$ and β is $p(x)$ for a unary predicate symbol p. Then, $A_p(\Gamma; \beta[v/x])$ is $A_p(p(v); p(v))$ which is equal to 1, while $\exists z A_p(\Gamma; \beta[z/x])$ is $\exists z A_p(p(v); p(z))$. But z is never equal to v as it is a variable which does not appear in $\{p(v), \exists x p(x)\}$. Therefore, $A_p(p(v); p(z))$ is $\bot$ and hence so does $\exists z A_p(p(v); p(z))$. Obviously, $\bot$ does not follow from 1.

When the language contains individual constants, an interpolant γ of a formula $\alpha \to \beta$ must satisfy the condition that not only predicate symbols but also individual constants in γ are contained in common in both α and β. For this purpose, it is necessary to introduce formulas $E_p^c(\Gamma)$ and $A_p^c(\Gamma; \alpha)$ such that neither predicate symbol p nor individual constant c appear in them. Now define $E_p^c(\Gamma)$ and $A_p^c(\Gamma; \alpha)$ by

- $E_p^c(\Gamma) \equiv \exists w(E_p(\Gamma)[w/c])$,
- $A_p^c(\Gamma; \alpha) \equiv \forall w(A_p(\Gamma; \alpha)[w/c])$.

Here, w is an individual variable not appearing in Γ for E_p^c and in Γ, α for A_p^c, respectively, and $E_p(\Gamma)[w/c]$ and $A_p(\Gamma; \alpha)[w/c]$ denote formulas obtained from $E_p^c(\Gamma)$ and $A_p^c(\Gamma; \alpha)$, respectively, by replacing every occurrence of c by w. Then we can show the following (2) and (3) in the same way as those in Theorem 7.3.

(1) (a) $\Gamma \Rightarrow E_p^c(\Gamma)$ is provable in $\mathbf{QFL_e}$,
    (b) $\vdash \Gamma, A_p^c(\Gamma; \alpha) \Rightarrow \alpha$ is provable in $\mathbf{QFL_e}$.

(2) Suppose that neither p nor c appear in Π. If $\Pi, \Gamma \Rightarrow \alpha$ is provable in $\mathbf{QFL_e}$ then
    (a) $\Pi, E_p^c(\Gamma) \Rightarrow \alpha$ is provable in $\mathbf{QFL_e}$ when neither p nor c appear in α,
    (b) $\Pi \Rightarrow A_p^c(\Gamma; \alpha)$ is provable in $\mathbf{QFL_e}$.

The same argument holds also for other predicate extension **QFL**, $\mathbf{QFL_{ew}}$, $\mathbf{QInFL_e}$ and $\mathbf{QInFL_{ew}}$ even when the language contains individual constants. We note that for **QFL**, "a side condition Y" must be understood as the provability of Y in **QFL**. As **QFL** is decidable as mentioned in Theorem 7.2, we can confirm that Lemma 5.1 holds also for **QFL**.

THEOREM 7.5. *The uniform interpolation property holds for* **QFL**, $\mathbf{QFL_e}$, $\mathbf{QFL_{ew}}$, $\mathbf{QInFL_e}$ *and* $\mathbf{QInFL_{ew}}$ *without function symbols.*

Example 2. At the end of Section 3, we discussed interpolants of a sequent $p, s, \neg(p \wedge q) \Rightarrow q \to (r \vee s)$ in $\mathbf{FL_{ew}}$. We showed that the post-interpolant of the formula $p \cdot s \cdot \neg(p \wedge q)$ with respect to the set $\{q, s\}$ (which in fact is equal to $E_p(p, s, \neg(p \wedge q))$) is $\neg q \wedge s$, while the pre-interpolant of $q \to (r \vee s)$ with respect to $\{q, s\}$ (which is equal to $A_r(\emptyset; q \to (r \vee s))$) is $q \to s$.

To make a comparison with it, we consider here interpolants of the sequent $p(u), s(u, v), \neg\exists y(p(y) \wedge q(u, y)) \Rightarrow \forall x q(x, x) \to \exists z(r(z) \vee s(u, z))$ in $\mathbf{QFL_{ew}}$, where u and v are mutually distinct variables. Note that this sequent is provable in $\mathbf{QFL_{ew}}$. First, consider the set $\mathcal{E}_p(p(u), s(u, v), \neg\exists y(p(y) \wedge q(u, y)))$. The computation goes similarly to that in Example 1. One of its conjuncts is the negation of the formula of the form $A_p(\Delta; \exists y(p(y) \wedge q(u, y)))$, where Δ is a nonempty submultiset of $\{p(u), s(u, v)\}$. For any such Δ, $\mathcal{A}_p(\Delta; \exists y(p(y) \wedge q(u, y)))$ contains

$$A_p(\Delta; p(u) \wedge q(u, u)) \vee A_p(\Delta; p(v) \wedge q(u, v)) \vee \exists y' A_p(\Delta; p(y') \wedge q(u, y'))$$

where y' is a new variable. As the second and third disjuncts are always equal to $\bot$, $A_p(\Delta; \exists y(p(y) \wedge q(u, y)))$ is shown to be equal to $q(u, u)$ and hence this first conjunct is equal to $\neg q(u, u)$. Another conjunct is expressed essentially as $E_p(p(u)) \cdot E_p(s(u, v)) \cdot E_p(\neg\exists y(p(y) \wedge q(u, y)))$, which is equal to $s(u, v)$. Thus, $E_p(p(u), s(u, v), \neg\exists y(p(y) \wedge q(u, y)))$ is $\neg q(u, u) \wedge s(u, v)$. On the other hand, it can be shown that $A_r(\emptyset; \forall x q(x, x) \to \exists z(r(z) \vee s(u, z)))$ is equal to $\forall x' q(x', x') \to \exists z' s(u, z')$.

§8. Acknowledgments.

We would like to thank an anonymous referee for her/his helpful comments and suggestions.

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