Preliminaries — First Order Predicate Logic

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Logic as a Formal System

- Truth can be formalized (partially....)
- We will start with First Order Predicate Logic

Symbols

Constants T, F — True and False

Variables x — Use lowercase, possible subscripts.

Predicates and Sets P — Uppercase letters.

Connectives $\vee \quad \wedge \quad \rightarrow \quad (\quad) \quad \neg \quad \forall \quad \exists \quad . \quad \in$

Sentences

Valid sentences include

$$T \quad F \quad x \quad P(x)$$

If E_1 and E_2 are valid sentences in FOPL, then so are...

$$\begin{array}{ccc} E_1 \wedge E_2 & E_1 \vee E_2 & \neg E_1 \\ (E_1) & E_1 \rightarrow E_2 & \\ \exists x.E_1 & \exists x \in S.E_1 & \\ \forall x.E_1 & \forall x \in S.E_1 & \end{array}$$

Where x is a variable, P is a predicate, and S is a set.

Rewriting

- Given a true logical statement, we can transform them or combine them to make more logical statements.
- The idea of rewriting terms to make simpler, equivalent terms is at the heart of programming.
- It is also a common proof technique.

Properties of $\wedge$ and $\vee$

	$\neg T = F$	$\neg F = T$
Identities	$x \wedge x = x$	$x \vee x = x$
	$x \wedge F = F$	$x \vee F = x$
	$x \wedge T = x$	$x \vee T = T$

Commutative

$$x \wedge y = y \wedge x$$
$$x \vee y = y \vee x$$

Associative

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z$$
$$x \vee (y \vee z) = (x \vee y) \vee z$$

Distributive

$$x \wedge (y \oplus z) = (x \wedge y) \oplus (x \wedge z)$$
$$x \vee (y \oplus z) = (x \vee y) \oplus (x \vee z)$$

De Morgan

$$\neg(x \wedge y) = \neg x \vee \neg y$$
$$\neg(x \vee y) = \neg x \wedge \neg y$$

Excluded Middle $\neg x \vee x = T$

Example Proof 1

Instead of simple variables, we will use some predicates from arithmetic.
Is it true that $\neg(\neg(x < 0) \wedge \neg(x > 0))$?

Start	$\neg(\neg(x < 0) \wedge \neg(x > 0))$
De Morgan	$((x < 0) \vee (x > 0))$
Definition of $\vee$ and $=$	$x \neq 0$

Each line of the proof should have the step and its justification. Note that equality has *three* cases!

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Exercise 1

- Prove that $\neg x \wedge x = F$.
- What will your strategy be? (Forward / Backward)

Start	$\neg x \wedge x = F$
Negate both sides	$\neg(\neg x \wedge x) = T$
De Morgan	$x \vee \neg x = T$
Excluded Middle	$T = T$

Exercise 1

- Prove that $\neg x \wedge x = F$.
- What will your strategy be? (Forward / Backward)

Start	$\neg x \wedge x = F$
Negate both sides	$\neg(\neg x \wedge x) = T$
De Morgan	$x \vee \neg x = T$
Excluded Middle	$T = T$

Implication

The symbol $\rightarrow$ is convenient, but completely unnecessary.

$$x \rightarrow y \equiv \neg x \vee y$$

"If x is true, then y is true. But if x is false, we know nothing about y."

Also, you should know about *Modus Ponens*...
$x \wedge (x \rightarrow y) \rightarrow y$

Example: All Chicago winters are cold.
Jan. 16 is in winter.
Therefore, Jan. 16 is cold.

Implication Example

If you skip the final, you will fail the course.
Suppose you take the final.
Is it true that you pass the course?

Let s be "skipped final", and let e be "failed course" (since f is already taken).

Then we want to solve $(\neg s \wedge (s \to e)) \to \neg e$.

Try to solve it.

.

Exercise 2

Start	$(\neg s \wedge (s \rightarrow e)) \rightarrow \neg e$
Substitute	$(T \wedge (F \rightarrow e)) \rightarrow \neg e$
Def of $\wedge$	$(F \rightarrow e) \rightarrow \neg e$
Def of $\rightarrow$	$(T \vee e) \rightarrow \neg e$
Def of $\vee$	$T \rightarrow \neg e$
Def of $\rightarrow$	$F \vee \neg e$
Def of $\vee$	$\neg e$

What does this tell us?

Exists and For All

We will use several notations.

- $\forall x \in S.P(x)$
- $\exists x \in S.P(x)$

If the domain of x is understood, we can write:

- $\forall x.P(x)$
- $\exists x.P(x)$

We can also treat a set like a predicate.

- $\forall x.S(x) \rightarrow P(x)$
- $\exists x.S(x) \wedge P(x)$

Be careful: the following mean different things than what we have already shown.

- $\exists x.S(x) \rightarrow P(x)$.
- $\forall x.S(x) \wedge P(x)$.

An important identity

$$\neg\exists x.\neg P(x) \quad \equiv \quad \forall x.P(x)$$

$$\neg\forall x.\neg P(x) \quad \equiv \quad \exists x.P(x)$$

Exercise 3

Write logical formulas for the following. Make up notation if you need to. We will introduce formal notation later.

- All apples are bad.
- Some apples are bad.
- Not all apples are bad.
- Some apples are not bad.
- Are the last two equivalent? Prove or disprove.
- There is an element in the array *A* that is greater than zero.
- Every student in the class scored more than 90% on the exam.

Expressing Bad Apples

- All apples are bad...
  - $\forall x.A(x) \rightarrow B(x)$
  - $\forall x \in A.B(x)$
  - Why not $\forall x.A(x) \wedge B(x)$?
- Some apples are bad...
  - $\exists x.A(x) \wedge B(x)$
  - $\exists x \in A.B(x)$
  - Why not $\exists x.A(x) \rightarrow B(x)$?
- Not all apples are bad.
  - $\neg\forall x.A(x) \rightarrow B(x)$
  - $\neg\forall x \in A.B(x)$
- Some apples are not bad.
  - $\exists x.A(x) \wedge \neg B(x)$
  - $\exists x \in A.\neg B(x)$

Not all apples are bad vs some apples are not bad.

Is "not all apples are bad" the same as saying "some apples are not bad"?

Start	$\neg\forall x.A(x) \rightarrow B(x)$
Negation	$\exists x.\neg(A(x) \rightarrow B(x))$
Def of $\rightarrow$	$\exists x.\neg(\neg A(x) \vee B(x))$
De Morgan	$\exists x.(A(x) \wedge \neg B(x))$

Not all apples are bad vs some apples are not bad.

Is "not all apples are bad" the same as saying "some apples are not bad"?

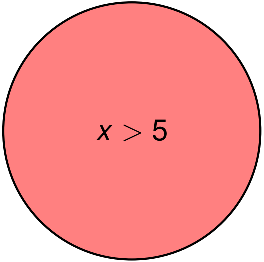
Start	$\neg\forall x.A(x) \to B(x)$
Negation	$\exists x.\neg(A(x) \to B(x))$
Def of $\to$	$\exists x.\neg(\neg A(x) \vee B(x))$
De Morgan	$\exists x.(A(x) \wedge \neg B(x))$

More Exercises

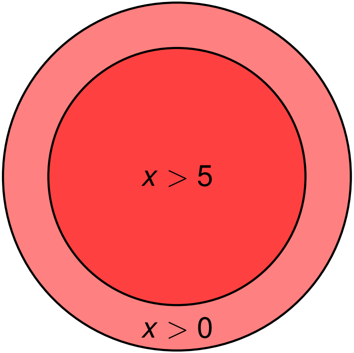
- There is an element in the array A that is greater than zero.
- Every student in the class scored more than 90% on the exam.

Formulas as Diagrams

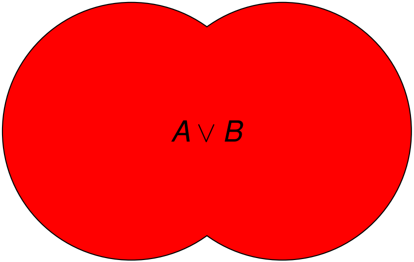
- A formula expresses true conditions.
- The variables in the formula create a coordinate range.



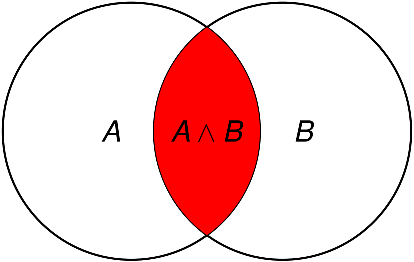
Formulas as Diagrams



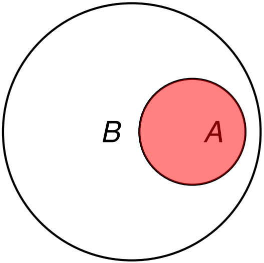
$A \vee B$



$A \wedge B$



$A \rightarrow B$



$A \rightarrow B$

Strength of a Predicate

- If $A \to B$, then A is said to be *stronger* than B, and B is said to be *weaker* than A.
- A is more restrictive than B—it will be true less often than B.
- This contrast will be *very* important later!
- Which is stronger? $x > 5$, or $x > 20$.
- Which is stronger? $A \wedge B$, or $A \vee B$?
- Which is weaker? A, or $A \vee B$?
- Which is weaker? A, or $A \wedge B$?
- Is A weaker than A? (no, not a typo.)

Weakening

You have many options if you want to weaken a predicate.

- Add a disjoint. (E.g., A becomes $A \vee B$.)
- Delete a conjunct. (E.g., $A \wedge B$ becomes A.)
- Replace a constant with a range. (E.g., $i = n$ becomes $0 \leq i \leq n$.)

Questions:

- What is the strongest possible predicate?
- What is the weakest possible predicate?
- Can you prove it?
- Is it possible to have two predicates A and B such that neither is weaker than the other?

Exercise 4

- When is $T \rightarrow x$ true?
- When is $F \rightarrow x$ true?
- Suppose $x \rightarrow y \wedge y \rightarrow z$. Suppose also $\neg x$. Can z be true? Must z be true?
- Suppose $\forall x.S(x) \rightarrow P(x)$. Which of the following are true?
  - $\exists x.S(X) \wedge P(X)$
  - $\exists x.\neg S(X) \wedge P(X)$
  - $\exists x.S(X) \wedge \neg P(X)$
  - $\exists x.\neg S(X) \wedge \neg P(X)$